JOURNAL OF ALGEBRA 31, 393–426 (1974)

On Rings with Central Polynomials

LOUIS HALLE ROWEN

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Communicated by N. Jacobson

Received August 6, 1973

Let Ω be a commutative ring with 1, and let R be an Ω-algebra with center C. A polynomial $f(X_1, ..., X_m)$ with coefficients in Ω is an *identity* of R if $f(r_1, ..., r_m) = 0$ for all $r_1, ..., r_m$ in R, and f is *central* in R if f is not an identity of R but $f(r_1, ..., r_m) \in C$ for all $r_1, ..., r_m$ in R. Formanek has shown recently that $M_n(\Omega)$, the algebra of $n \times n$ matrices with entries in Ω, has a central polynomial which is linear in the last variable. Call such a central polynomial *regular*. It follows from Formanek's result that a large class of algebras have regular central polynomials. The purpose of this paper is to examine ideals of rings with regular central polynomial and to see what relationships exist between these ideals and ideals of the center. In order to do this it is necessary to develop some general results on identities of algebras and to investigate *central localization*, i.e., localization of R by a multiplicative subset S of C.

As an application of the results of Sections 1–4, we obtain some information on semiprime P.I.-algebras in Section 5 and conclude with a fairly straightforward proof in Section 6 of the Artin–Procesi theorem, which says that a ring R (with 1) is Azumaya of rank n^2 over its center if and only if R satisfies the identities of $M_n(\mathbb{Z})$ and no homomorphic image of R satisfies the identities of $M_{n-1}(\mathbb{Z})$.

1. INTRODUCTION: IDENTITIES OF (ASSOCIATIVE) ALGEBRAS

Throughout the body of this paper all rings are associative with multiplicative unit. Let Ω be a commutative ring. We shall consider the category of associative Ω-algebra with 1, hereafter called *algebras*. (Homomorphisms will mean algebra homomorphisms mapping 1 into 1, and subalgebras will contain 1.)

Let $\Omega\{X\} \equiv \Omega\{X_1, X_2, ...\}$ be the free (associative) algebra generated by a countable set of noncommuting indeterminates. $\Omega\{X\}$ is also free as Ω-module,

* This paper was written while the author was a doctoral student at Yale University, under the guidance of Nathan Jacobson to whom the author is deeply grateful. Partial support was provided by National Science Grant No. NSF-GP-33591.

with countable base consisting of 1 and the distinct monomials $X_{i_1} \cdots X_{i_s}$ where $s \geqslant 1$ and $i_j = 1, 2, \ldots$. Given any algebra R and a countable set of elements $r_1, r_2, \ldots$ in R, we get a unique homomorphism of $\Omega\{X\}$ into R sending $X_i \to r_i$ for all i.

The elements of $\Omega\{X\}$ will be called *polynomials*; if $f \in \Omega\{X\}$ is contained in the subalgebra generated by $X_1, \ldots, X_m$ then we denote f as $f(X_1, \ldots, X_m)$. The image in R of f under the homomorphism sending $X_i \to r_i$, $1 \leqslant i \leqslant m$, is written $f(r_1, \ldots, r_m)$, and f is an *identity* of R if $f(r_1, \ldots, r_m) = 0$ for all choices of $r_1, \ldots, r_m$ in R, i.e., if f is in the kernel of every homomorphism from $\Omega\{X\}$ to R. Note that any identity has constant term 0, immediately seen by setting $r_1 = r_2 = \cdots = r_m = 0$.

Writing a polynomial f as a unique linear combination of elements of the base of $\Omega\{X\}$, we call the monomials (with nonzero coefficients) in this expression the *monomials of* f. We say f is *linear* in X_i if every monomial of f is of degree 1 in X_1 ; f is *multilinear* if f is linear in every X_i occuring in f (that is, X_i having positive degree in some monomial of f). Similarly, f is *homogeneous in* X_i if all monomials of f have the same degree in X_i ; f is *completely homogeneous* if f is homogeneous in every X_i. Similarly, f is *blended in* X_i if X_i occurs in every monomial of f; f is *blended* if f is blended in every X_i occurring in f. Evidently all completely homogeneous polynomials are blended.

Generalizing [14, p. 28], we define the *height* of a monomial to be its degree minus the number of indeterminates occurring in the monomial, and the height of a polynomial f (denoted as $ht\, f$) to be the maximum height of the monomials of f. It is apparent that f is multilinear if and only if f is blended and has height 0.

An element $g(X_1, \ldots, X_m)$ of $\Omega\{X\}$ is *central for* R if $[X_{m+1}, g] \equiv X_{m+1}g - gX_{m+1}$ is an identity of R and if g is not an identity of R. In other words, $g(r_1, \ldots, r_m) \in C$, the center of R, all $r_1, \ldots, r_m$ in R and there exist $r_1, \ldots, r_m$ in R such that $g(r_1, \ldots, r_m) \neq 0$. Clearly any two algebras satisfying the same identities also have the same central polynomials.

If $\bar{G}$ is a subgroup of the additive group of an algebra R, then $f \in \Omega\{X\}$ is called *$\bar{G}$-valued for* R if $f(r_1, \ldots, r_m) \in \bar{G}$ for all choices of the r_i in R. If $\bar{G} = 0$ then all $\bar{G}$-valued polynomials are identities, and if $\bar{G} = C$ then all $\bar{G}$-valued polynomials which are not identities are central. A polynomial $f(X_1, \ldots, X_m)$ is *additive in* X_i on R, X_i occurring in f, if for all $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_m$ in R the map $r_i \mapsto f(r_1, \ldots, r_i, \ldots, r_m)$ is additive; f is *additive on* R if f is additive in X_i on R for every X_i occurring in f.

LEMMA 1.1. *Any polynomial f is a sum of blended polynomials f_j, each having the following properties*:

(1) $ht\, f_j \leqslant ht\, f$;

(2) *f_j is linear in every indeterminate in which f is linear;*

(3) *for any algebra R and any additive subgroup $\tilde{G}$ of R, f_j is $\tilde{G}$-valued on R if f is $\tilde{G}$-valued on R.*

Proof. If f is blended, then there is nothing to prove. Otherwise let $t > 0$ be the number of indeterminates in which f is not blended. The proof is by induction on this number. For simplicity, we may assume f is not blended in X_1. Then $f'(X_2, ..., X_m) \equiv f(0, X_2, ..., X_m)$ is the sum of all monomials of f in which X_1 does not occur, and $f'' \equiv f - f'$ is blended in X_1. Since f' and f'' have $< t$ indeterminates in which they are not blended, the result holds by induction for f' and f''. Thus it holds for $f = f' + f''$. Q.E.D.

Suppose X_1 occurs in f. As in the usual linearization process [13, p. 225], define

$$\begin{aligned} \Delta f(X_1, ..., X_{m+1}) \equiv\ & f(X_1 + X_{m+1}, X_2, ..., X_m) \\ & - f(X_1, X_2, ..., X_m) \\ & - f(X_{m+1}, X_2, ..., X_m). \end{aligned}$$

Δf is an identity of R if and only if f is additive in X_1 on R, and if f is blended then Δf is blended and $ht\, \Delta f < ht\, f$.

LEMMA 1.2. *Let f be a polynomial which is not an identity of a given algebra R. Then there exists a polynomial $\hat{f}$ with the following properties:*

(1) *$ht\, \hat{f} \leqslant ht\, f$;*

(2) *$\hat{f}$ is linear in every indeterminate in which f is linear;*

(3) *$\hat{f}$ is additive and is not an identity on R;*

(4) *For any algebra R' and additive subgroup $\tilde{G}'$ of R', if f is $\tilde{G}'$-valued then $\hat{f}$ is $\tilde{G}'$-valued;*

(5) *if f is blended but not additive on R, then $\hat{f}$ is blended and $ht\, \hat{f} < ht\, f$.*

Proof. By Lemma 1.1 we may assume f is blended. If f is additive on R then we are done, so assume f is not additive in X_1. Then $\Delta f(X_1, ..., X_{m+1})$ is not an identity on R and $ht\, \Delta f < ht\, f$. Moreover, Δf is linear in every indeterminate in which f is linear, and for every algebra R' and additive subgroup $\tilde{G}'$ of R', Δf is $\tilde{G}'$-valued if f is $\tilde{G}'$-valued. Since any blended polynomial of height 0 is multilinear, hence additive on any algebra, (1)–(4) hold for Δf by induction on height. Thus (1)–(4) hold for $\hat{f}$, and (5) follows from the fact $ht\, \Delta f < ht\, f$. Q.E.D.

Now consider the following situation: R and R_1 are algebras such that

R_1 satisfies the identities of R, R_1 is also an algebra over a commutative ring K, and H is a commutative K-algebra. The algebra conditions imply R_1 is an Ω-K bimodule, so $R' = R_1 \otimes_K H$ is an Ω-algebra with the operation $\omega(\Sigma r_i \otimes h_i) = \Sigma \omega r_i \otimes h_i$ for all r_i in R_1, h_i in H.

PROPOSITION 1.3. (i) *If f is a completely homogeneous identity of R which is additive on R' then f is an identity of R'.*

(ii) *If each identity f of R is the sum of completely homogeneous identities, each of height $\leqslant ht\, f$, then R' satisfies the identities of R.*

Proof. (i) Since f is additive on R', it suffices to show

$$f(r_1 \otimes h_1, \ldots, r_m \otimes h_m) = 0$$

in R' for all $r_1, \ldots, r_m$ in R_1, $h_1, \ldots, h_m$ in H. Let the degree of the ith indeterminate of f be d_i, $1 \leqslant i \leqslant m$. Then $f(r_1 \otimes h_1, \ldots, r_m \otimes h_m) = f(r_1, \ldots, r_m) \otimes h_1^{d_1} \cdots h_m^{d_m} = 0$, as desired.

(ii) Suppose R' does not satisfy all the identities of R. Choose a polynomial f of minimal height such that f is an identity of R but not of R'. By hypothesis we may assume f is completely homogeneous. If f is additive on R', then f is an identity of R' by (i), a contradiction. But if f is not additive on R', we note that since f is 0-valued on R, Lemma 1.2 can be applied (interchanging the roles of R and R') to obtain a polynomial $\hat{f}$ which is not an identity of R' but which is 0-valued on R, hence an identity of R; moreover $ht\, \hat{f} < ht\, f$, a contradiction to the minimality of $ht\, f$. Either way we have a contradiction, so R' satisfies the identities of R. Q.E.D.

Several trivial observations are in order. First, Proposition 1.3(i) implies any multilinear identity of R is an identity of R', a well known fact.

Conversely, it is immediate whenever $R \hookrightarrow R'$ that R satisfies the identities of R'.

Remark 1.4. Suppose R is an algebra and Ω contains an infinite subdomain Ω' such that $\omega' r \neq 0$ for every nonzero ω' in Ω' and r in R. A well-known application of the Vandermonde determinant argument shows that any identity f of R is a sum of completely homogeneous identities, each of height $\leqslant ht\, f$. Thus, the following three sentences are immediate from Proposition 1.3.

(1) Let R be an algebra, and assume Ω contains an infinite subdomain Ω' such that $\omega' r \neq 0$ for every $\omega' \neq 0$ in Ω' and $r \neq 0$ in R. If R is also an algebra over a commutative ring K and if H is a commutative K-algebra, then $R \otimes_K H$ is an Ω-algebra satisfying the identities of R.

(2) Let R be a ring satisfying the identities of $M_n(\mathbb{Z})$. If R is also an algebra over a commutative ring K and if H is a commutative K-algebra then $R \otimes_K H$ satisfies the identities of $M_n(\mathbb{Z})$.

(3) If Ω is a field and R is central simple (over Ω) of finite degree n, then R and $M_n(\Omega)$ satisfy the same identities. This is well known and immediate. If Ω is finite then $R \approx M_n(\Omega)$ by Wedderburn's theorem on finite division rings; if Ω is infinite then we get the result by splitting R by the algebraic closure of Ω.

2. P.I.-Algebras and Formanek Polynomials

The purpose of most of this section is to give a general definition of P.I.-algebra over an arbitrary commutative ring Ω, which preserves the key features of the classical theory. At the end of the section we will introduce Formanek's polynomials and state some of their known properties.

We begin by stating several facts about (polynomial) identities, each well known before 1955. The *standard polynomial on k letters* is $S_k(X_1, \ldots, X_k) = \Sigma(sg\pi)X_{\pi 1} \cdots X_{\pi k}$, the sum taken over all permutations π of $(1, \ldots, k)$. Obviously S_k is multilinear of degree k and alternating. Amitsur–Levitzki [6] showed S_{2n} is an identity of $M_n(\Omega)$, minimal in the sense that any multilinear identity of degree $\leqslant 2n$ is of the form ωS_{2n}, ω in Ω; hence, S_{2n} is a minimal identity of any simple algebra of dimension n^2 over its center.

Let R be an Ω-algebra with center C. In [2] Amitsur calls a polynomial *proper* (*for* R) if at least one of its coefficients does not annihilate R. Suppose f is a proper identity of R (in particular $f \neq 0$). We can write $f = f_1 + f_2$ where each coefficient of f_1 does not annihilate R and each coefficient of f_2 annihilates R. But then each monomial of f_2 is an identity of R, so f_2 is an identity of R; hence, f_1 is an identity of R, none of whose coefficients annihilates R. The usual linearization procedure yields a multilinear identity of R, none of whose coefficients annihilates R. In particular, any algebra with a proper identity has a proper multilinear identity.

Now, for any element f in $\Omega\{X\}$, let Ω_f denote the set of coefficients (in Ω) of monomials of f and let $\Omega_f R$ be the additive subgroup of R generated by all ωr, ω in Ω_f, r in R. In other words, $\Omega_f R = \{\Sigma \omega_i r_i \mid \omega_i \in \Omega_f, r_i \in R\}$. Note that $\Omega_f R$ is an ideal of R, and $\Omega_f R \neq 0$ if and only if f is proper.

The starting point for the general theory is Amitsur's extension of Kaplansky's theorem for P.I.-algebras to the following statement [1]: A primitive algebra R satisfies a proper identity of degree d if and only if R is simple of dimension $\leqslant [d/2]^2$ over its center, where $[x]$ denotes the greatest integer in x.

Hence, a primitive algebra satisfying a proper identity actually satisfies a standard identity (and is simple).

Recall that R is *prime* if the product of two nonzero ideals is nonzero, and R is *semiprime* if every nilpotent ideal is 0. R is semiprime if and only if R is the subdirect product of prime algebras; similarly an algebra is called *semi-primitive* (*semisimple*) if it is the subdirect product of primitive (simple) algebras. A key result follows.

THEOREM 2.1 (Amitsur [13, p. 12]). *If R has no nil ideals, then $R[\lambda] = R \otimes_C C[\lambda]$ is semiprimitive, where λ is a commutative indeterminate.*

Note if R satisfies a proper identity then $R[\lambda]$ satisfies a proper multilinear identity (for R satisfies a proper multilinear identity, and $R[\lambda]$ satisfies the same identity by Proposition 1.3(i)). This enables Amitsur to prove the following proposition.

PROPOSITION 2.2 (Amitsur [2]). *Let R be prime and let f be a proper identity of R of degree d. Then $S_{2[d/2]}$ is an identity of R.*

Our structure theory is based on the following strengthening of the notion of proper identity.

DEFINITION 2.3. An algebra R is a *P.I.-algebra* if there exists an identity of R which is proper for every nonzero homomorphic image of R. A *P.I.-ring* is a P.I.-Z-algebra.

Clearly any homomorphic image of a P.I.-algebra is a P.I.-algebra.

By Kaplansky's theorem every primitive image of a P.I.-algebra is simple; hence, every semiprimitive P.I.-algebra is semisimple.

Define the *degree* of R to be the smallest n such that $S_{2n}(X_1, \ldots, X_{2n})^m$ is an identity of R for some m. In view of Proposition 2.2, it is clear that every semiprime P.I.-algebra R has finite degree, and if $n = \deg R$ then S_{2n} is an identity of R (and of $R[\lambda]$, which also has degree n). Also observe (by the Amitsur–Levitzki theorem) that the degree of a central simple algebra is the same as its degree as a P.I.-algebra.

Suppose R is a P.I.-algebra with lower nilradical N (cf. [13, p. 194]). Then R/N is a semiprime P.I.-algebra. Hence R/N satisfies a standard identity, implying R/N has no nil ideals (cf. [13, p. 232]). Therefore the lower and upper nilradicals of R are the same, and will be called the *nilradical*.

Remark 2.4. R is a P.I.-algebra if and only if there exists an identity f of R such that $\Omega_f R = R$. It follows that any $R' = \prod_\alpha R$, a complete direct product of copies of R, is a P.I.-algebra.

Proof. ($\Rightarrow$) Let f be an identity of R which is proper for every nonzero homomorphic image of R. Since f is not proper for $R/\Omega_f R$, we must have $R/\Omega_f R = 0$, so $\Omega_f R = R$.

($\Leftarrow$) Suppose f is an identity of R such that $\Omega_f R = R$. Then $\Omega_f \bar{R} = \bar{R}$ for any nonzero homomorphic image $\bar{R}$ of R. In particular f is proper for any nonzero homomorphic image $\bar{R}$, so R is a P.I.-algebra.

The final assertion follows from $1 \in R$. Q.E.D.

In light of Remark 2.4 we can reformulate a recent result of Amitsur [4].

THEOREM 2.5. (*Amitsur*). *Any* P.I.-*algebra satisfies an identity* $(S_{2n}(X_1, ..., X_{2n}))^m$ *for suitable* n *and* m.

Proof (Amitsur). Choose an identity f of a P.I.-algebra R such that $\Omega_f R = R$. Let $\deg f = d$, $n = [d/2]$, let α range over all $2n$-tuples of elements of R and let $R' = \prod_\alpha R$ (complete direct product of copies R_α of R). Clearly R' is a P.I.-algebra, and if N' is the nilradical of R then R'/N' is a semiprime P.I.-algebra of degree $\leqslant n$. Hence $S_{2n}(h_1, ..., h_{2n}) \in N'$ for all $h_1, ..., h_{2n}$ in R'.

Now choose h_i, $1 \leqslant i \leqslant 2n$, such that $h_i(\alpha) = r_i$ if $\alpha = (r_1, ..., r_{2n})$. Since N' is nil we have m such that $S_{2n}(h_1, ..., h_{2n})^m = 0$. Then

$$0 = S_{2n}(h_1(\alpha), ..., h_{2n}(\alpha))^m = S_{2n}(r_1, ..., r_{2n})^m$$

for all $\alpha = (r_1, ..., r_{2n})$, so S_{2n}^m is an identity for R.

This shows that any P.I.-algebra is in fact a P.I.-ring in the classical sense, i.e., satisfying an identity with coefficients in $\mathbb{Z}$, one of whose coefficients is ± 1. In fact all coefficients of S_{2n}^m are ± 1. Multilinearizing this identity yields the following corollary in light of Proposition 1.3(i).

COROLLARY 2.6. *Suppose R is a* P.I.-*algebra, and let R also be a K-algebra, K an arbitrary commutative ring. For any commutative K-algebra H, $R \otimes_K H$ is a* P.I.-*algebra.*

Another immediate corollary of Proposition 1.9 is that a subalgebra of a P.I.-algebra is a P.I.-algebra.

Let us now consider a special class of polynomials, namely those central for $M_n(\Omega)$. In a beautifully direct construction, Formanek [12] (cf. also Amitsur [5] for exposition and extensions) solved a 25-year old question by finding, for any given n, a polynomial $g_n(X_1, ..., X_{n+1})$ (with constant term 0) which is central for $M_n(\Omega)$, Ω arbitrary, g_n having the following additional properties:

(i) coefficients in $\mathbb{Z}$, one of which is ± 1.

(ii) complete homogeneity.

(iii) linearity in each indeterminate except X_1.

(iv) degree n^2.

It is a well known consequence of Remark 1.4 that g_n is central for any central simple algebra of degree n.

Amitsur [5] has obtained a very useful application of Formanek's work.

THEOREM 2.7 (*Amitsur*). *Let* $g_{0n} = g_n$ *be the Formanek polynomial for* $M_n(\Omega)$. *Then there exist central polynomials*

$$g_{1n}(X_1, \ldots, X_{n+1}), \ldots, g_{nn}(X_1, \ldots, X_{n+1})$$

for $M_n(\Omega)$ *satisfying conditions* (i), (ii) *and* (iii) *such that for any*

$r_1, r_2, \ldots, r_{n+1} \in M_n(\Omega)$,

$$\begin{aligned} g_{0n}(r_1, \ldots, r_{n+1})\, \lambda^n &+ g_{1n}(r_1, \ldots, r_{n+1}) \lambda^{n-1} \\ &+ \cdots + g_{nn}(r_1, \ldots, r_{n+1}) \\ &= g_{0n}(r_1, \ldots, r_{n+1}) f(\lambda, r_1), \end{aligned}$$

where $f(\lambda, r_1)$ *is the characteristic polynomial of* r_1. *Thus, by the Hamilton–Cayley theorem,* $g_{0n}(X_1, \ldots, X_{n+1}) X_1{}^n + g_{1n}(X_1, \ldots, X_{n+1}) X_1^{n-1} + \cdots + g_{nn}(X_1, \ldots, X_{n+1})$ *is an identity for* $M_n(\Omega)$.

Remark 2.8. Procesi has made the following neat observation: Any central polynomial of $M_n(\Omega)$ is an identity of $M_{n-1}(\Omega)$.

Proof. Let $\{e_{ij} \mid 1 \leqslant i, j < n\}$ be the set of matric units for $M_n(\Omega)$ and view $M_{n-1}(\Omega) \subseteq M_n(\Omega)$ with matric units $\{e_{ij} \mid 1 \leqslant i, j \leqslant n-1\}$. Then for $g(X_1, \ldots, X_m)$ central for $M_n(\Omega)$ and $r_1, \ldots, r_m$ in $M_{n-1}(\Omega)$, $e_{nn} g(r_1, \ldots, r_m) = 0$. Hence $g(r_1, \ldots, r_m) = 0$. Q.E.D.

An easy corollary of Procesi's observation follows.

COROLLARY 2.9. *If* Ω *is an infinite field and if* R *and* $M_n(\Omega)$ *satisfy the same identities, then for any semiprime ideal* B *of* R, *either* R/B *satisfies the same identities as* R *or every central polynomial of* R *is an identity of* R/B.

Proof. R/B is a semiprime P.I.-algebra over an infinite field, thus satisfying the same identities as $M_t(\Omega)$ for some t. If $t = n$ then R and R/B satisfy the same identities; otherwise we are done by Remark 2.8.

A major tool in the body of this paper is the following application of Formanek's central polynomials.

THEOREM 2.10 (*Rowen* [17, *Theorem* 2]). *Any nonzero ideal of a semiprime* P.I.-*algebra intersects the center nontrivially.*

COROLLARY 2.11. *If the center of a semiprime* P.I.-*algebra* R *is a field, then* R *is simple.*

Proof. Immediate from Theorem 2.10.

3. Central Localization

In this section we develop a natural generalization of commutative localization of arbitrary algebras. The interest of the results is considerably increased when central localization is applied to algebras with central polynomial.

Let R be an algebra (over the commutative ring Ω) with center C. Viewing C as a *monoid* (i.e., multiplicative set with 1), let S be a submonoid of C and consider $R \times S = \{(r, s) | r \in R, s \in S\}$. $R \times S$ has a relation $\sim$, given by $(r_1, s_1) \sim (r_2, s_2)$ if and only if $(r_1 s_2 - r_2 s_1)s = 0$ for some s in S. This relation is easily seen to be an equivalence. Let rs^{-1} denote the equivalence class of (r, s) and let R_S be the set of these equivalence classes. R_S can be given the well defined operations

$$\begin{aligned} r_1 s_1^{-1} + r_2 s_2^{-1} &= (r_1 s_2 + r_2 s_1)(s_1 s_2)^{-1}, \\ (r_1 s_1^{-1})(r_2 s_2^{-1}) &= (r_1 r_2)(s_1 s_2)^{-1}, \\ \omega(rs^{-1}) &= (\omega r)\, s^{-1} \text{ for } \omega \text{ in } \Omega,\ rs^{-1} \text{ in } R_S. \end{aligned}$$

Endowed with these operations, R_S becomes an algebra called *the localization of R by S*. This process of localization by submonoids of C will be called *central localization.*

There is a canonical homomorphism $\nu_S : R \to R_S$ given by $r \to r1^{-1}$ and for all s in S, $s1^{-1}$ has inverse $1s^{-1}$. Clearly $\ker \nu_S = \{r \in R \mid rs = 0$ for some s in $S\}$. This provides characterization of central localization as follows: Let R' be any algebra such that there is a homomorphism α: $R \to R'$ with $\alpha(s)$ invertible in R' for all s in S. Then there is a unique homomorphism β: $R_S \to R'$ making the following diagram commutative:

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\nu_S"'] & R' \\ R_S \arrow[ur, "\beta"'] & \end{tikzcd}$$

where β is given by $\beta(rs^{-1}) = \alpha(r)\, \alpha(s)^{-1}$. Clearly $\ker \beta = \{rs^{-1} \mid r \in \ker \alpha\}$, so β is a monomorphism if α is a monomorphism.

Suppose S' and S are submonoids of C, and $S' \subseteq S$. Since for every s' in S', $\nu_S(s')$ is invertible in R_S, we have a canonical homomorphism β: $R_{S'} \to R_S$ as above. Assume, moreover, that every $\nu_{S'}(s)$, s in S, is invertible in $R_{S'}$. Then we have β': $R_S \to R_{S'}$ which is an inverse to β, so β is an isomorphism.

An interesting special case of central localization occurs when every element s of S is *regular* (i.e., for all nonzero r in R, $sr = rs \neq 0$). In this case $r_1 s_1^{-1} =$

$r_2 s_2^{-1}$ if and only if $r_1 s_2 = r_2 s_1$, so the canonical map ν_S is injective. Let T be the set of all regular central elements. R_T is called the ***algebra of central quotients*** of R. If for all s in T, $s1^{-1}$ is invertible in R_S, then we have seen that R_S is isomorphic to the algebra of central quotients of R since $S \subseteq T$. Note that if R is prime then every nonzero element of C is regular.

(In general, we can pass to the above special case by noting $R_S \approx R_{S'}$, where $R' = \nu_S(R)$ and $S' = \nu_S(S)$, and each element of S' is regular in R'.) Also, it is clear (since S is commutative) that any set of m elements in R_S can be put in the form $r_1 s^{-1}, r_2 s^{-1}, \ldots, r_m s^{-1}$ for suitable $r_1, \ldots, r_m$ in R, s in S.

THEOREM 3.1. *R_S satisfies all the identities of R. If every element of S is regular then R satisfies the identities of R_S.*

Proof. (a) Let f be an identity of R. In view of the remark preceding this theorem, it suffices to show $f(r_1 s^{-1}, \ldots, r_m s^{-1}) = 0$ for $r_1, \ldots, r_m$ in R and s in S. Let $f_t(X_1, \ldots, X_m)$ be the sum of all monomials of f which have (total) degree t, and let $y_t = f_t(r_1, \ldots, r_m)$. If f has degree d then for $0 \leqslant j \leqslant d$,

$$\sum_{t=0}^{d} s^{jt} y_t = f(s^j r_1, \ldots, s^j r_m) = 0.$$

Using the Vandermonde determinant argument on this system of $d + 1$ equations (with y_t as the variables, $0 \leqslant t \leqslant d$), we get $h(s) y_t = 0$ for all t, where the Vandermonde determinant $h(s)$ is a product of terms of the form $s^p - s^q$, $p < q$. Evidently $h(s)$ is of the form $s^k h'(s)$ where $h'(s)$ is a polynomial in s with integral coefficients and constant term 1. The canonical map from R to R_S gives $h(s) y_t 1^{-1} = 0$; $h'(s) y_t 1^{-1} = 0$ is obtained by multiplying through by $1 s^{-k}$. Let $h'(s) = 1 - s\, h''(s)$. Then (for all t) $(1 - s h''(s)) y_t 1^{-1} = 0$, so $y_t 1^{-1} = s h''(s) y_t 1^{-1}$. Therefore,

$$\begin{aligned} f(r_1 s^{-1}, \ldots, r_m s^{-1}) &= \sum_{t=0}^{d} f_t(r_1 s^{-1}, \ldots, r_m s^{-1}) = \sum_{t=0}^{d} y_t s^{-t} \\ &= \sum_{t=0}^{d} (h''(s))^t y_t 1^{-1} = \sum_{t=0}^{d} f_t(h''(s)\, r_1, \ldots, h''(s)\, r_m)\, 1^{-1} \\ &= f(h''(s)\, r_1, \ldots, h''(s)\, r_m)\, 1^{-1} = 0, \end{aligned}$$

so indeed f is an identity on R_S. Q.E.D.

Conversely, if all elements of S are regular, then $R \hookrightarrow R_S$ as algebras, so it is trivial that any identity of R_S is an identity of R.

LEMMA 3.2. cent $R_S \supseteq C_S$, *equality holding if all elements of S are regular.*

Proof. Let $c_1s_1^{-1} \in C_S$, c_1 in C, s_1 in S. For any rs^{-1} in R_S, r in R and s in S, $(c_1s_1^{-1})(rs^{-1}) = (c_1r)(s_1s)^{-1} = (rc_1)(ss_1)^{-1} = (rs^{-1})(c_1s_1^{-1})$, so $\text{cent}(R_S) \supseteq C_S$.

Conversely, suppose that all elements of S are regular, and let $r_1s_1^{-1} \in \text{cent}(R_S)$ for r_1 in R, s_1 in S. For any r in R, $(r_1s_1^{-1})(r1^{-1}) = (r1^{-1})(r_1s_1^{-1})$, so $(r_1r)s_1^{-1} = (rr_1)s_1^{-1}$. In other words, $(rr_1 - r_1r)s_1 = 0$, so $r_1r - rr_1 = 0$, implying $\text{cent}(R_S) = C_S$. Q.E.D.

The next result is a strengthening of Posner's theorem, discovered independently, using different methods, by Formanek, Martindale, Procesi, Schacher, Small, and Rowen [17].

THEOREM 3.3. *Let R be a prime P.I.-algebra of degree n. Then the algebra R' of central quotients of R is simple of degree n (hence of dimension n^2 over its center, which is the quotient field of C). Moreover, R' and R satisfy the same identities and R' is the classical left and right algebra of quotients of R.*

Proof. By Theorem 3.1, R' satisfies the same identities as R, and cent R' is the quotient field of C by Lemma 3.2. Hence, by Corollary 2.11, R' is simple and clearly has degree n. In particular, R' is left and right artinian, so R' is the classical left and right quotient algebra of R. Q.E.D.

As in commutative localization, attention focuses on prime ideals P of C. Let R_P denote the central localization of R with respect to $C - P$. The interest of studying R_P is increased by the fact that Zorn's lemma implies the existence of an ideal $\tilde{B}$ of R, maximal with respect to $\tilde{B} \cap C \subseteq B$, for any given ideal B of C. The most fundamental connections between these ideals are in the following lemma.

LEMMA 3.4. (a) *Let $\tilde{A}$ be an ideal in R. Then $A = \tilde{A} \cap C$ is an ideal in C and A is proper if and only if $\tilde{A}$ is proper. If $\tilde{A}$ is prime (semiprime) then A is prime (semiprime).*

(b) *Let B be an ideal in C, and let $\tilde{B}$ be any ideal of R maximal with respect to the property $\tilde{B} \cap C \subseteq B$. Then $\tilde{B}$ is proper if and only if B is proper. If B is prime (semiprime), then $\tilde{B}$ is prime (semiprime).*

Proof. (a) Clearly A is an ideal in C. Trivially A is proper $\Leftrightarrow 1 \notin A \Leftrightarrow 1 \notin \tilde{A} \Leftrightarrow \tilde{A}$ is proper. Suppose $\tilde{A}$ is prime. Let $a, b \in C$ such that $ab \in A$. Then $aRb \subseteq \tilde{A}$, so $a \in \tilde{A} \cap C = A$ or $b \in A$. Hence A is prime. Likewise, suppose $\tilde{A}$ is semiprime and let $a \in C$ such that $a^2 \in A$. Then $aRa \subseteq \tilde{A}$, so $a \in A$, showing that A is semiprime.

(b) $\tilde{B}$ is proper $\Leftrightarrow 1 \notin \tilde{B} \Leftrightarrow 1 \notin B \Leftrightarrow B$ is proper. Suppose B is prime. Let $\tilde{A}_1$, $\tilde{A}_2$ be ideals of R containing $\tilde{B}$, with $\tilde{A}_1\tilde{A}_2 \subseteq \tilde{B}$. Let $A_1 = \tilde{A}_1 \cap C$, $A_2 = \tilde{A}_2 \cap C$. Clearly $A_1A_2 \subseteq B$, so $A_1 \subseteq B$ or $A_2 \subseteq B$. Assume $A_1 \subseteq B$. Then by definition of $\tilde{B}$, $\tilde{A}_1 \subseteq \tilde{B}$. However $\tilde{A}_1 \supseteq \tilde{B}$, so $\tilde{A}_1 = \tilde{B}$ and $\tilde{B}$ is prime. The analogous argument shows $\tilde{B}$ is semiprime if B is semiprime. Q.E.D.

LEMMA 3.5. (i) *R prime $\Rightarrow R_P$ is prime for all prime ideals P of C.*

(ii) *R_P semiprime for all maximal ideals P of $C \Rightarrow R$ is semiprime.*

Proof. (i) Suppose R_P is not prime for some prime ideal P of C. Then there exist nonzero r_1, r_2 in R, s_1, s_2 in $S = C - P$, with $(r_1s_1^{-1})(r1^{-1})(r_2s_2^{-1}) = 0$ for all r in R. Since R is prime, its nonzero central elements are regular, so $r_1rr_2 = 0$, implying $r_1Rr_2 = 0$, a contradiction since $r_1 \neq 0$, $r_2 \neq 0$. Q.E.D.

(ii) Suppose R is not semiprime, i.e., for some nonzero r_1 in R, $r_1rr_1 = 0$ for all r in R. Let $B = \{c \in C \mid cr_1 = 0\}$. Clearly B is a proper ideal of C contained in a maximal ideal P of C. In R_P we have for all r in R, all s in $S = C - P$, $(r_11^{-1})(rs^{-1})(r_11^{-1}) = 0$. Since R_P is semiprime, $r_1s_1 = 0$ for some s_1 in S. But $s_1 \in B \cap S = \varnothing$, a contradiction, showing that R is semiprime. Q.E.D.

Given a subset $\tilde{A}$ of R and a prime ideal P of C, we define $\tilde{A}_P = \{rs^{-1} \mid r \in \tilde{A}, s \in S = C - P\} \subseteq R_P$. If $\tilde{A}$ is an ideal in R then $\tilde{A}_P$ is an ideal in R_P.

LEMMA 3.6. *Let P be a prime ideal of C, and let $\tilde{P}$ be an ideal of R maximal with respect to $\tilde{P} \cap C \subseteq P$. Then $P \subseteq \tilde{P} \Leftrightarrow P_P \subseteq \tilde{P}_P$.*

Proof. Trivially $P \subseteq \tilde{P}$ implies $P_P \subseteq \tilde{P}_P$. To show the converse, let $c \in P$. Then $c1^{-1} \in P_P$. If $P_P \subseteq \tilde{P}_P$, then there exist p in $\tilde{P}$ and s in $S = C - P$ such that $c1^{-1} = ps^{-1}$. Thus, for some s_1 in S, $(cs - p)s_1 = 0$. Hence $css_1 \in \tilde{P}$, so $c \in \tilde{P}$. This proves $P \subseteq \tilde{P}$. Q.E.D.

THEOREM 3.7. *Let P be a prime ideal of C, and let $\tilde{P}$ be an ideal of R maximal with respect to $\tilde{P} \cap C \subseteq P$.*

(i) *$\tilde{P}_P$ is maximal in R_P.*

(ii) *Let $\bar{R} = R/\tilde{P}$, $S = C - P$, $\bar{S} = (S + \tilde{P})/\tilde{P}$. Then there is a canonical embedding $\bar{R} \hookrightarrow R_P/\tilde{P}_P$ which induces an isomorphism $R_P/\tilde{P}_P \approx \bar{R}_{\bar{S}}$, which is the algebra of central quotients of $\bar{R}$. In particular, $\tilde{P}$ is maximal in R, if and only if* $\operatorname{cent}(R/\tilde{P})$ *is a field, in which case $R/\tilde{P} \approx R_P/\tilde{P}_P$.*

Proof. (i) First we claim $\tilde{P}_P$ is proper. Otherwise $1 \in \tilde{P}_P$, implying there are p_1 in $\tilde{P}$, s_1 in $S = C - P$, such that $p_1s_1^{-1} = 1$, that is, $(p_1 - s_1)s = 0$ for some s in S. Then $s_1s \in \tilde{P}$. But $s_1s \in S$ and $S \cap \tilde{P} = \varnothing$, a contradiction, so $\tilde{P}_P$ is proper. Now let A' be any ideal of R_P such that $\tilde{P}_P \subset A'$. We claim $1 \in A'$.

Indeed, there exist r_1 in R, s_1 in S, such that $r_1s_1^{-1} \in A' - \tilde{P}_P$. Clearly

$r_1 \notin \tilde{P}$, but $r_1 1^{-1} = (r_1 s_1^{-1})(s_1 1^{-1}) \in A'$. Let $\tilde{A} = \{r \in R \mid r1^{-1} \in A'\}$. $\tilde{A}$ is an ideal of R containing both $\tilde{P}$ and r_1, so by definition of $\tilde{P}$, $\tilde{A} \cap C \not\subseteq P$. Let $s_3 \in \tilde{A} \cap S = \tilde{A} \cap (C - P) \neq \varnothing$. Then $1 = s_3 s_3^{-1} \in A'$, implying $\tilde{P}_P$ is maximal in R_P.

(ii) Define Ψ: $R \to R_P/\tilde{P}_P$ by $\Psi(r) = r1^{-1} + \tilde{P}_P$. Clearly $\tilde{P} \subseteq \ker \Psi$. We claim $\tilde{P} = \ker \Psi$. Well for any r in R, $r \in \ker \Psi$ if and only if $r1^{-1} \in \tilde{P}_P$, which means for some p_1 in $\tilde{P}$, s_1 in S, $r1^{-1} = p_1 s_1^{-1}$; so for some s in S, $(rs_1 - p_1)s = 0$. Thus $rs_1 \in \tilde{P}$; since $\tilde{P}$ is prime, $r \in \tilde{P}$. Therefore $\tilde{P} = \ker \Psi$, and we have a canonical monomorphism $\bar{R} \hookrightarrow R_P/\tilde{P}_P$, where $\bar{R} = R/\tilde{P}$. Let $\bar{S} = \{s + \tilde{P} \mid s \in S\}$. Clearly $0 \notin \bar{S}$ and $\bar{S}$ is a submonoid of cent $\bar{R}$. Moreover for $s \in S$, $\Psi(s) \in \operatorname{cent}(R_P/\tilde{P}_P)$, a field since $R_P/\tilde{P}_P$ is simple. Thus in the injection $\bar{R} \hookrightarrow R_P/\tilde{P}_P$, the image of each element of $\bar{S}$ is invertible, so $\bar{R}_{\bar{S}} \hookrightarrow R_P/\tilde{P}_P$ is induced canonically. On the other hand, for any $r_1 s_1^{-1}$ in R_P, $r_1 s_1^{-1} + \tilde{P}_P$ is the image of $(r_1 + \tilde{P})(s_1 + \tilde{P})^{-1}$, an element of $\bar{R}_{\bar{S}}$; so $\bar{R}_{\bar{S}} \hookrightarrow R_P/\tilde{P}_P$ is an isomorphism. In particular $\bar{R}_{\bar{S}}$ is simple, so every nonzero element of cent $\bar{R}$ is invertible in $\bar{R}_{\bar{S}}$ (via the injection $\bar{R} \hookrightarrow \bar{R}_{\bar{S}}$). This implies $\bar{R}_{\bar{S}}$ is the ring of central quotients of $\bar{R}$, which is isomorphic to $R_P/\tilde{P}_P$.

Now $\bar{R}$ is its algebra of central quotients if and only if cent $\bar{R}$ is a field, in which case $\bar{R} \approx R_P/\tilde{P}_P$ which is simple, so $\tilde{P}$ is maximal in R. On the other hand, if $\tilde{P}$ is maximal in R then cent $\bar{R}$ is a field. Q.E.D.

A question considered in Section 6 is whether there are classes of algebras for which the classical algebra of left quotients exists and is the algebra of central quotients. Clearly all that one needs to verify (cf. [13, p. 261]) is that for all x regular in R, $x1^{-1}$ is invertible in R', the algebra of central quotients of R. One situation for which this condition is easy and well known is when R' is left artinian. The condition is left–right symmetric, so if the left classical algebra of left quotients exists and is the algebra of central quotients, then the classical algebra of right quotients also exists and is the algebra of central quotients.

4. Structure of Algebras with Central Polynomial

In this section we investigate the structural implications of the existence of *regular* central polynomials on an algebra R (not assumed a priori to be a P.I.-algebra).

Definition 4.1. A polynomial g is *regular* if g is linear in one of its indeterminates.

By rearranging the indeterminates, we may assume that a regular polynomial is linear in its last indeterminate.

EXAMPLE 4.2. The Formanek polynomial g_n is regular; hence g_n is a regular central polynomial for any semiprime P.I.-algebra of degree n (as is well known and easily verified using P.I. structure theory).

EXAMPLE 4.3. In [13, p. 260] it is shown that $[X_1, X_2] = X_1X_2 - X_2X_1$ is central for all Grassman algebras.

EXAMPLE 4.4. Any Lie nilpotent algebra of nilpotence degree n has the regular central polynomial $[[\cdots[[X_1, X_2], X_3],...], X_{n-1}]$.

Let I_g be the set of values (in C) taken by a central polynomial g of R. The motivation for studying regular central polynomials arises from the following.

Remark 4.5. If g is regular, then I_g is a nonzero monoid ideal of C. Consequently, $I_g = C$ if and only if some element of I_g is invertible.

Proof. For $r_1,..., r_m$ in R, c in C, $g(r_1,..., r_m)c = g(r_1,..., r_{m-1}, rc) \in I_g$; the remark follows immediately.

Let $g(X_1,..., X_m)$ be a central polynomial of R. Clearly some coefficient of g does not annihilate R, so $[g(X_1,..., X_m), X_{m+1}]$ is a proper identity of R. Moreover, if $g(r_1,..., r_m)$ is invertible for some choice of $r_1,..., r_m$ in R then this is the case for every nonzero homomorphic image of R; hence $[g, X_{m+1}]$ is a proper identity for every nonzero homomorphic image of R, so R is a P.I.-algebra.

Let $g(X_1,..., X_m)$ be a regular central polynomial of R and suppose $g(r_1,..., r_m)$ is invertible. If r in R commutes with $r_1,..., r_m$ then $r \in C$. Indeed, let $c = g(r_1,..., r_m)^{-1}$. Then $c \in C$. Since g is linear in the last indeterminate, $r = g(r_1,..., r_m)\, cr = g(r_1,..., r_m cr) \in C$, as claimed. These facts are used in proof of the following theorem.

THEOREM 4.6. *Let R have a regular central polynomial. If C is a field then R is simple and finite dimensional over C.*

Proof. Let $g(X_1,..., X_m)$ be a regular central polynomial of R. Since C is a field, $I_g = C$ by Remark 4.5. Hence R is a P.I.-algebra, and we need show only that R is simple (by Kaplansky's theorem).

Let $\bar{N}$ be the nilradical of R, let $\bar{R} = R/\bar{N}$, and let $\bar{I}_g = (I_g + \bar{N})/\bar{N}$, a field. By Remark 4.5, $\bar{I}_g = \text{cent}\, \bar{R}$, so $C = I_g \approx \bar{I}_g = \text{cent}\, \bar{R}$ and we identify C with cent $\bar{R}$. By Corollary 2.11, $\bar{R}$ is simple. Let $\bar{R}$ have degree n.

Let $F = C$ if C is finite and let F be the algebraic closure of C if C is infinite. $M_n(F) \approx \bar{R} \otimes_C F$ satisfies the same identities as $\bar{R}$, so g is central for $M_n(F)$. Setting $\tilde{G}' = C$, Lemma 2.1 (with $M_n(F)$ and R replacing respectively R and R') yields a regular polynomial $\hat{g}$, additive and central for $M_n(F)$, hence central for $\bar{R}$. Now let $R' = R \otimes_C F$, $N' = \tilde{N} \otimes_C F$. Since $\tilde{N}$ is locally nilpotent, N' is a nil ideal of R' and is proper since $1 \notin N'$. But the canonical homomorphism $R \otimes_C F \to \bar{R} \otimes_C F$ has kernel N', so N' is maximal in R' (since $\bar{R} \otimes_C F$ is simple).

Hence N' is the unique prime ideal of R', which means R' is *primary* (cf. [13, p. 56]) with nil Jacobson radical N'. By [13, p. 54], R' is S.B.I. (suitable for building idempotents), implying by [13, p. 56] $R' = M_n(H)$ where H is an F-algebra and $H/\text{rad}\, H = F$.

Since $\hat{g}$ is central and additive in $M_n(F)$, there exist matric units $x_1, \ldots, x_t$ in $M_n(F)$ and $\alpha, \alpha_1, \ldots, \alpha_t$ in F such that $\hat{g}(\alpha_1 x_1, \ldots, \alpha_t x_t) = \alpha \neq 0$. But $\hat{g}$ is regular central in R'; since any h in H commutes with $\alpha_1, \ldots, \alpha_t$ and the matric units $x_1, \ldots, x_t$, we conclude $H \subseteq \text{cent}\, R' = F$. Hence $H = F$, implying $R' = M_n(F)$ and R is simple. Q.E.D.

COROLLARY 4.7. *Let R be an algebra with regular central polynomial. If every nonzero element of C is regular then R is prime.*

Proof. The algebra of central quotients of R satisfies the conditions of Theorem 4.6 and is thereby simple. Hence R is prime. Q.E.D.

If R is a semiprime P.I.-algebra then some Formanek polynomial is central for R. In this way we see Theorem 4.6 generalizes Corollary 2.11.

EXAMPLE 4.8. Let Ω be a domain and let R_1 be the algebra of upper triangular matrices in $M_2(\Omega)$. Every element of cent R_1 is regular although R_1 is not semiprime; an instant application of Corollary 4.7 shows R_1 has no regular central polynomial, a fact which is easily verified directly.

EXAMPLE 4.9. Let R_1 be as in the previous example, and let $R = R_1 \oplus M_2(\Omega)$. R_1 has the ideal $\tilde{B} = \{\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right) \mid \alpha \in \Omega\}$ which has trivial intersection with cent R_1, so $\tilde{B} \oplus 0$ has trivial intersection with cent R. Since the Formanek polynomial g_2 is central for R, we see that Theorem 2.10 cannot be generalized for algebras with regular central polynomial in the way Theorem 4.6 generalized Corollary 2.11.

DEFINITION 4.10. The *central kernel* I of an algebra R is the additive subgroup of C generated by all the I_g (for all regular central polynomials g of R). If R has no regular central polynomials we set $I = 0$.

Remark. I is an ideal of the commutative ring C.

LEMMA 4.11. *Let S be a submonoid of C. $I_S \subseteq$ central kernel of R_S, and I_S is an ideal of* cent R_S.

Proof. Let $g(r_1, \ldots, r_m)s^{-1} \in I_S$, g a regular central polynomial of R. Then $g(r_1 1^{-1}, \ldots, r_{m-1} 1^{-1}, r_m s^{-1}) = g(r_1, \ldots, r_m)s^{-1}$, so, to show $I_S \subseteq$ central kernel of R_S, we need only show that any regular central polynomial $g(X_1, \ldots, X_m)$ of R is central or an identity for R_S. This is immediate by Theorem 3.1.

Now let $g(r_1, \ldots, r_m)s_1^{-1} \in I_S$ and $rs^{-1} \in$ cent R_S. Then

$$\begin{aligned}(g(r_1, \ldots, r_m)\, s_1^{-1})\, rs^{-1} &= g(r_1 1^{-1}, \ldots, r_m 1^{-1})(1 s_1^{-1})(rs^{-1}) \\ &= g(r_1 1^{-1}, \ldots, r_{m-1} 1^{-1}, r_m rs^{-1})(1 s_1^{-1}) \\ &= g(r 1^{-1}, \ldots, r_{m-1} 1^{-1}, r_m r 1^{-1})(1 s^{-1})(1 s_1^{-1}) \\ &= g(r_1, \ldots, r_{m-1}, r_m r)(ss_1)^{-1} \in I_S,\end{aligned}$$

so I_S is an ideal of cent R_S. Q.E.D.

DEFINITION 4.12. An ideal $\tilde{B}$ of R is *identity-faithful* if there is a regular polynomial central both for R and for $R/\tilde{B}$.

Clearly $\tilde{B}$ is identity-faithful if and only if $\tilde{B} \not\supseteq I_g$ for some regular central polynomial g, if and only if $\tilde{B} \not\supseteq I$, if and only if $\tilde{B} \cap C \not\supseteq I$. This motivates the following definition.

DEFINITION 4.13. An ideal B of C is *identity-faithful* if $B \not\supseteq I$.

LEMMA 4.14. *Let $\tilde{P}$ be an identity-faithful maximal ideal of R and let $P = \tilde{P} \cap C$. Then* cent $R/\tilde{P} = (C + \tilde{P})/\tilde{P}$ *and P is maximal in C.*

Proof. Let $\bar{R} = R/\tilde{P}$ and let g be a regular central polynomial of R such that $I_g \not\subseteq \tilde{P}$. Then $0 \neq \bar{I}_g = (I_g + \tilde{P})/\tilde{P} \subseteq$ cent $\bar{R}$, a field since $\bar{R}$ is simple. Let $F =$ cent $\bar{R}$. $F \supseteq (C + \tilde{P})/\tilde{P} \supseteq \bar{I}_g = F$ since $\bar{I}_g$ is a monoid ideal of F, so $F = (C + \tilde{P})/\tilde{P} \approx C/P$, implying P is maximal in C. Q.E.D.

LEMMA 4.15. *Let $\tilde{P}$ be an identity-faithful prime of R and let $P = \tilde{P} \cap C$. For any ideal $\tilde{A}$ of R such that $\tilde{A} \cap C \subseteq P$, we have $\tilde{A} \subseteq \tilde{P}$.*

Proof. Let $\bar{R} = R/\tilde{P}$, let $\bar{A} = (\tilde{A} + \tilde{P})/\tilde{P}$, and let $A' = \bar{A} \cap$ cent $\bar{R}$. Since $I \not\subseteq \tilde{P}$, there is a regular polynomial $g(X_1, \ldots, X_m)$, central both for R and for $\bar{R}$; in particular, $[g, X_{m+1}]$ is a proper identity for $\bar{R}$, so $\bar{R}$ is a P.I.-algebra by Proposition 2.2. Let $g(\bar{r}_1, \ldots, \bar{r}_m) \neq 0$. For any $\bar{a}$ in A', $g(\bar{r}_1, \ldots, \bar{r}_m)\bar{a} = \overline{g(r_1, \ldots, r_m a)} \subseteq \overline{C \cap \tilde{A}} \subseteq \bar{P} = 0$; hence $A' = 0$ (since $\bar{R}$ is prime), so $\bar{A} = 0$ by Theorem 2.10. Therefore $\tilde{A} \subseteq \tilde{P}$. Q.E.D.

Our main result on identity-faithful ideals follows:

THEOREM 4.16. *Let P be an identity-faithful prime ideal of C, and let $\tilde{P}$ be an ideal of R maximal with respect to $\tilde{P} \cap C \subseteq P$. Then $\tilde{P}$ is prime and*

(a) $\operatorname{cent}(R_P) = C_P =$ *central kernel of* $R_P = I_P$.

(b) $P \subseteq \tilde{P}$; *in other words* $P = \tilde{P} \cap C$.

(c) *$\tilde{P}$ is the union of all ideals of R whose intersection with C is P. In particular, $\tilde{P}$ is the only prime ideal of R such that $\tilde{P} \cap C = P$, yielding a one-to-one order-preserving correspondence $P \leftrightarrow \tilde{P}$ between identity-faithful prime ideals of C and identity-faithful prime ideals of R.*

(d) *If P is maximal then $\tilde{P}$ is maximal; hence the correspondence given in* (c) *yields a canonical one-to-one order-preserving correspondence $P \leftrightarrow \tilde{P}$ between identity-faithful maximal ideals of C and identity-faithful maximal ideals of R. Moreover in this case $PR = \tilde{P}$.*

(e) *For any identity-faithful prime ideal $\tilde{P}$ of R, and for $P = \tilde{P} \cap C$, $R_P/\tilde{P}_P$ is isomorphic to the classical algebra of quotients of $R/\tilde{P}$.*

Proof. We note $\tilde{P}$ is prime by Lemma 3.4(b). Also $\tilde{P}$ is identity-faithful. Let $S = C - P$.

(a) Since $I \not\subseteq P$, $I \cap S = \varnothing$. Let $s \in I \cap S$. Then $1 = ss^{-1} \in I_P$. But I_P is an ideal of cent R_P by Lemma 4.11. Hence $I_P = C_P =$ cent R_P. Moreover, letting I' be the central kernel of R_P, we have $I_P \subseteq I' \subseteq \operatorname{cent} R_P = I_P$, so $I' = I_P$.

(b) By (a), $C_P = \operatorname{cent} R_P$ and $\tilde{P}_P$ is identity-faithful in R_P. Moreover, $\tilde{P}_P$ is maximal in R_P by Theorem 3.7(i) so $\tilde{P}_P \cap C_P$ is maximal in C_P by Lemma 4.14. We claim $\tilde{P}_P \cap C_P \subseteq P_P$, which would imply $\tilde{P}_P \cap C_P = P_P$ since $P_P \neq C_P$. Well, if $x \in \tilde{P}_P \cap C_P$ then $x = c_1 s_1^{-1} = p_2 s_2^{-1}$ for suitable c_1 in C, s_1, s_2 in S, p_2 in $\tilde{P}$. Then for some s in S, $(c_1 s_2 - p_2 s_1)s = 0$, implying $c_1 \in \tilde{P} \cap C \subseteq P$, establishing the claim. But $\tilde{P}_P \cap C_P = P_P \Rightarrow \tilde{P} \cap C = P$ by Lemma 3.6.

(c) If $\tilde{A}$ is any ideal such that $\tilde{A} \cap C = P$, then $\tilde{A} \subseteq \tilde{P}$ by Lemma 4.15. On the other hand, if $\tilde{P}_1$ is prime and $\tilde{P}_1 \cap C = P$, then $\tilde{P} \subseteq \tilde{P}_1$ by Lemma 4.15, so $\tilde{P} = \tilde{P}_1$.

(d) If $\tilde{P}$ were not maximal, then we would have an ideal $\tilde{A}$ with $\tilde{P} \subset \tilde{A} \subset R$. But then $P \subset \tilde{A} \cap C \subset C$, contrary to P maximal in C. To prove in fact that PR is maximal, we observe first that $P \subseteq PR \cap C \subseteq \tilde{P} \cap C = P$, so $P = PR \cap C$. Let $\bar{I} = (I + PR)/PR \approx I/(I \cap PR) = I/I \cap P \neq 0$ and let $\bar{C} = (C + PR)/PR \approx C/P$, a field. $\bar{I} \subseteq \bar{C}$ so $\bar{I} = \operatorname{cent} \bar{R}$ by Remark 4.5. Hence cent $\bar{R} = \bar{C}$, a field, implying $\bar{R}$ is simple by Theorem 4.6, i.e., PR is maximal in R. The rest of this part follows from Lemma 4.14. (Note that Theorem 4.6 was used in only proving $PR = \tilde{P}$.)

(e) Since $I \not\subseteq P$, $R/\tilde{P}$ has a proper identity, hence a proper multilinear identity. By (c), $\tilde{P}$ is maximal with respect to $\tilde{P} \cap C \subseteq P$. Thus by Theorem 3.7(i) the algebra of central quotients of $R/\tilde{P}$ is $R_P/\tilde{P}_P$, a simple algebra with a proper identity, hence simple artinian and the classical algebra of quotients of $R/\tilde{P}$. Q.E.D.

Theorem 4.16(e) extends work by Small [19] in which he considered non-commutative localization of a prime P.I.-algebra R by a prime ideal $\tilde{P}$ such that R and $R/\tilde{P}$ had the same degree.

EXAMPLE 4.17. Let λ be a commutative indeterminate, p a prime, and let $\Omega = \mathbb{Z}[\lambda]/(\lambda^2\mathbb{Z}[\lambda] + p\lambda\mathbb{Z}[\lambda])$. In other words letting $x \mapsto \bar{x}$ in the canonical homomorphism $\mathbb{Z}[\lambda] \to \Omega$, we have $\bar{\lambda}^2 = 0$ and $\bar{p}\bar{\lambda} = 0$. Let R be the subalgebra of $M_2(\Omega)$ whose entries along the diagonal are in $\bar{\mathbb{Z}} \cong \mathbb{Z}$ and whose entries off the diagonal are in $\bar{\lambda}\bar{\mathbb{Z}} + \bar{p}\bar{\mathbb{Z}}$.

Suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in C = \operatorname{cent} R$. Then

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix},$$

so $b = c = 0$. It follows that $C = \bar{\mathbb{Z}} \cdot \left(\begin{smallmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{smallmatrix}\right) = \bar{\mathbb{Z}}$, identifying as usual the identity matrix of R as 1. It is easy to see $I \neq 0$ since the Formanek polynomial g_2 is central for R (obvious because there is an embedding $M_2(p\mathbb{Z}) \hookrightarrow R$). On the other hand, 0 is a prime ideal of C but certainly not of R because R has nilradical

$$\left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \middle| \, b, c \in \bar{\lambda}\bar{\mathbb{Z}}\right\},$$

as is verified directly.

This example shows that P prime in C does not imply PR prime in R, even if P is identity-faithful. In particular one cannot say in a ring with regular central polynomial that C prime $\Rightarrow$ R prime. There is a weaker result, however, which is of some consolation.

LEMMA 4.18. *Let $I \neq 0$. If $P = 0$ is a prime ideal of C then R_P is simple and there is a unique prime ideal $\tilde{P}$ of R such that $\tilde{P} \cap C = 0$ and $\tilde{P}_P = 0$.*

Proof. Since $P \not\supseteq I$, there is a unique prime ideal $\tilde{P}$ of R such that $\tilde{P} \cap C = P = 0$, by Theorem 4.16(c). By Theorem 4.16(a), $I_P = \operatorname{cent} R_P = C_P =$ the field of quotients of the domain C; hence R_P is simple by Theorem 4.6. $\tilde{P}_P$ is proper (by Theorem 3.7), so $\tilde{P}_P = 0$. Q.E.D.

PROPOSITION 4.19. *Let P be a prime ideal of C and let $\tilde{P}$ be maximal with respect to $\tilde{P} \cap C \subseteq P$.* If *$\tilde{P}$ is identity-faithful then P is identity-faithful.*

Proof. Suppose $\tilde{P}$ is identity-faithful, i.e., $I \not\subseteq \tilde{P}$, and let $\bar{R} = R/\tilde{P}$, $\bar{I} = (I + \tilde{P})/\tilde{P} \neq 0$. The embedding $\bar{R} \hookrightarrow R_P/\tilde{P}_P$ (of Theorem 3.7(ii)), given by $r + \tilde{P} \mapsto r \cdot 1^{-1} + \tilde{P}_P$, shows that there exists a regular central polynomial g of R, and $r_1, \ldots, r_m$ in R, such that $g(r_1 1^{-1}, \ldots, r_m 1^{-1}) \notin \tilde{P}_P$; hence $I_P \not\subseteq \tilde{P}_P$. Since g is central for R_P, by Theorem 3.1, we get $\tilde{P}_P$ is identity-faithful in R_P. Let $C' = \operatorname{cent} R_P$ and let $P' = \tilde{P}_P \cap C'$. By Theorem 3.7(ii), $\tilde{P}_P$ is maximal in R_P, so P' is maximal in C' by Lemma 4.14; by Lemma 4.11, I_P is an ideal of C', implying $I_P + P' = C'$. Let $cs_1^{-1} + ps_2^{-1} = 1$ for suitable c in I, p in $\tilde{P}$, s_1, s_2 in S, ps_2^{-1} in P'. Then $(cs_2 + ps_1 - s_1 s_2)s = 0$ for some s in S, so $ps_1 s = s_1 s_2 s - cs_2 s \in C$. But then $ps_1 s \in \tilde{P} \cap C \subseteq P$ and $s_1 s_2 s \in S = C - P$, implying $cs_2 s \notin P$. Since $cs_2 s \in I$, we conclude $I \not\subseteq P$, which means P is identity-faithful. Q.E.D.

COROLLARY 4.20. $I = C \Leftrightarrow IR = R$.

Proof. If $I = C$ then $1 \in I$, implying $IR = R$. Inversely, if $I \neq C$ then $I \subseteq P$ for some prime ideal P of C. Let $\tilde{P}$ be an ideal of R maximal with respect to $\tilde{P} \cap C \subseteq P$. If $\tilde{P} \cap C \not\supseteq I$ then $\tilde{P}$ is identity-faithful, so P is identity-faithful by Proposition 4.19, contradicting $I \subseteq P$. Hence $\tilde{P} \cap C \supseteq I$, so $IR \subseteq \tilde{P}R = \tilde{P} \neq R$.

This concludes the basic elements of the theory of the central kernel. It is appropriate to mention that Procesi [35] has concurrently obtained results about the Formanek center of a semiprime P.I.-ring of degree n, which is the ideal generated by the values taken by all central polynomials. The approaches are quite different; for one thing, he assumes the Artin–Procesi theorem, which will actually be seen to be a consequence of the results of this section (cf. Section 6); on the other hand, he has much interesting geometric insight, whereas the focus here is strictly ring-theoretic.

We close this section by examining some important special cases of Theorem 4.16, closely related to Azumaya algebras.

DEFINITION 4.21. *R is identity-faithful if every proper ideal of R is identity-faithful.*

THEOREM 4.22.(a) *R is identity-faithful if and only if $I = C$, in which case R is a P.I.-algebra.*

(b) *Let P be an identity-faithful prime of C. Then R_P is identity-faithful and has unique maximal ideal $(PR)_P$. $(PR)_P \cap C_P = P_P$.*

(c) *Let $\bar{C} = (C + \tilde{B})/\tilde{B}$, $\tilde{B}$ and ideal of R, let $\bar{I} = (I + \tilde{B})/\tilde{B}$, and let $\bar{R} = R/\tilde{B}$. If R is identity-faithful or if $\bar{R}$ is identity-faithful with central kernel $\bar{I}$, then $\bar{C} = $* cent $\bar{R}$.

Proof. (a) Clearly R is identity-faithful if and only if $IR = R$, which happens if and only if $I = C$, by Corollary 4.10. In this case some central polynomial takes on the value 1, implying R is a P.I.-algebra.

(b) By Theorem 4.16(a), $C_P = \operatorname{cent} R_P =$ central kernel of R_P, implying R_P is identity-faithful. Since P_P is the unique maximal ideal of C_P, we have from Theorem 4.16(d) that $(PR)_P = P_P R_P$ is the unique maximal ideal of R_P, and $(PR)_P \cap C_P = P_P$.

(c) Clearly $\bar{I} \subseteq \bar{C} \subseteq \operatorname{cent} \bar{R}$, so it suffices to prove $\bar{I} = \operatorname{cent} \bar{R}$. If R is identity-faithful then $1 \in \bar{I}$, so this follows trivially from the fact $\bar{I}$ is an ideal of cent $\bar{R}$. If $\bar{R}$ is identity-faithful with central kernel $\bar{I}$ then again we get $1 \in \bar{I}$, so $\bar{I} = \operatorname{cent} \bar{R}$. Q.E.D.

THEOREM 4.23. *Assume R is identity-faithful, and let B be a semiprime ideal of C. Then there exists a semiprime ideal $\tilde{B}$ of R such that $\tilde{B} \cap C = B$. Choose such an ideal $\tilde{B}$. For any ideal $\tilde{A}$ of R such that $\tilde{A} \cap C \subseteq B$, we have $\tilde{A} \subseteq \tilde{B}$. Hence $\tilde{B} = \bigcup$ {ideals of R whose intersection with C is B}, yielding a one-to-one order-preserving correspondence $B \leftrightarrow \tilde{B}$ of semiprime ideals of C and semiprime ideals of R.*

Proof. Suppose $B = \bigcap_i P_i$ for a suitable set of primes P_i of C. By Theorem 4.16, for each P_i there is a prime $\tilde{P}_i$ with $\tilde{P}_i \cap C = P_i$; $\tilde{B} = \bigcap_i \tilde{P}_i$ is a semiprime ideal of R such that $\tilde{B} \cap C = B$. Suppose $\tilde{B}$ is any semiprime ideal of R such that $\tilde{B} \cap C = B$. Then $\tilde{B} = \bigcap_j \tilde{P}_j$, for suitable prime $\tilde{P}_j$ in R, and $B = \bigcap_j P_j$ where $P_j = \tilde{P}_j \cap C$. if $\tilde{A} \cap C \subseteq B$ then $\tilde{A} \cap C \subseteq P_j$ for each j, so $\tilde{A} \subseteq \tilde{P}_j$ by Lemma 4.15. Hence $\tilde{A} \subseteq \bigcap_j \tilde{P}_j = \tilde{B}$. The rest of the theorem follows easily by symmetry. Q.E.D.

One can obtain partial information even when I is a maximal ideal of C (such a situation arises in infinitely generated algebras of generic matrices, cf. Procesi [15], and has been treated in considerable depth by Small). First note in this situation that every ideal of C, other than I, is identity-faithful. Moreover, $IR \neq R$ by Corollary 4.20, so there exist prime ideals of R containing I.

THEOREM 4.24. *Suppose the central kernel I of R is maximal. Given a prime ideal $\tilde{P}$ of R with $I \subseteq \tilde{P}$, one has a canonical lattice injection $\beta_{\tilde{P}}$ from the set of semiprime ideals of C into the set of semiprime ideals of R, such that $\beta_{\tilde{P}}$ sends prime ideals of C to prime ideals of R.*

Proof. Consider a typical semiprime ideal B of C, and let $\mathscr{P} =$ {prime ideals of C containing B}. For each P_i in $\mathscr{P} - \{I\}$, let $\tilde{P}_i$ be the prime of R such that $\tilde{P}_i \cap C = P_i$ ($\tilde{P}_i$ exists and is unique by Theorem 4.16), and let $\tilde{B}_0 = \bigcap\{\tilde{P}_i \mid P_i \in \mathscr{P} - \{I\}\}$. If $I \notin \mathscr{P}$, let $\tilde{B} = \tilde{B}_0$; if $I \in \mathscr{P}$, let $\tilde{B} = \tilde{B}_0 \cap \tilde{P}$.

We claim $B \mapsto \tilde{B}$ is the desired correspondence $\beta_{\mathfrak{p}}$. Indeed, as in the proof of Theorem 4.23, one sees for any ideal $\tilde{A}$ of R such that $\tilde{A} \cap C \subseteq B$, we have $\tilde{A} \subseteq \tilde{B}_0$. Hence $\beta_{\mathfrak{p}}$ is a lattice isomorphism; if $B = I$ then $\tilde{B} = \tilde{P}$ is prime, and if $B \neq I$ is prime then $\tilde{B}$ is the ideal of R maximal with respect to $\tilde{B} \cap C = B$, so $\tilde{B}$ is prime. Q.E.D.

5. Application 1: Semiprime P.I.-Algebras

In this section let R be a semiprime P.I.-algebra (with center C). For the most part, the only results needed will be Theorem 2.10 and the beginning of Section 3, and the basic procedure will be to obtain information on R by looking at its center C. Some easy examples follow.

R is simple if and only if C is a field (by Corollary 2.11). R is prime if and only if C is prime (suppose C is prime. If A and B are ideals of R and if $AB = 0$, then $(A \cap C)(B \cap C) = 0$, so $A \cap C = 0$ or $B \cap C = 0$; hence $A = 0$ or $B = 0$ by Theorem 2.10).

We say r is *left (right) regular* in R if $r_1 r \neq 0$ ($rr_1 \neq 0$) for any nonzero r_1 in R. An element both left and right regular is *regular*.

Lemma 5.1. *If c is regular in C then c is regular in R.*

Proof. Suppose c is not regular in R. Then $cr = 0$ for some nonzero r in R; hence $cRrR = 0$, so $c(RrR \cap C) = 0$. But $(RrR) \cap C \neq 0$ by Theorem 2.10, so $cc_1 = 0$ for nonzero c_1 in $(RrR) \cap C$. Q.E.D.

Proposition 5.2. *If R' is the algebra of central quotients of R then* cent R' *is the classical algebra of quotients of C.*

Proof. Immediate from Lemmas 3.2 and 5.1.

Suppose an algebra R has a classical left (right) algebra of quotients R'. Then we say R is a *left (right) order* in R'.

Let $\operatorname{Ann} A = \{r \in R \mid ra = 0 \text{ for all } a \text{ in } A\}$, that is, $\operatorname{Ann} A$ is the *left annihilator* of A. If A is an ideal then $\operatorname{Ann} A$ is an ideal. In a semiprime ring the left annihilator and right annihilator are the same ($(A \operatorname{Ann} A)^2 = A(\operatorname{Ann} A)\, A(\operatorname{Ann} A) = 0$, so $A \operatorname{Ann} A = 0$; hence $\operatorname{Ann} A$ is contained in the right annihilator of A, and equality holds by symmetry).

If $B = \operatorname{Ann} A$ for some ideal A in R, then B is called an *annihilator in R*. Also, let $\operatorname{Ann}_C A = C \cap \operatorname{Ann} A$, and call $\operatorname{Ann}_C A$ an *annihilator in C*, if A is an ideal in C.

Lemma 5.3. (i) *If A is an ideal of C and if $B = \operatorname{Ann}_C A$ then $B = \operatorname{Ann}_C AR$.*

(ii) *For any ideal A of R,* $\operatorname{Ann} A = \operatorname{Ann}(A \cap C)$.

Proof. (i) Since $BAR = 0$, $B \subseteq C \cap \operatorname{Ann} AR = \operatorname{Ann}_C AR$. But $A \operatorname{Ann}_C AR = 0$, so $\operatorname{Ann}_C AR \subseteq \operatorname{Ann}_C A = B$.

(ii) Clearly $\operatorname{Ann} A \subseteq \operatorname{Ann}(A \cap C)$. On the other hand, suppose $(\operatorname{Ann}(A \cap C))A \neq 0$. Then by Theorem 2.10 there exists nonzero $c = \Sigma x_i a_i$ with c in C, x_i in $\operatorname{Ann}(A \cap C)$, a_i in A. But $c^2 \in (\operatorname{Ann}(A \cap C))(A \cap C) = 0$, contrary to R being semiprime. Hence $\operatorname{Ann} A = \operatorname{Ann}(A \cap C)$. Q.E.D.

It is very easy to show that any commutative semiprime algebra satisfying the ascending chain condition on annihilators of ideals is an order in a semisimple artinian algebra. This yields a result announced in [18], and obtained independently by S. Steinberg.

THEOREM 5.4. (i) *If C is an order in a semisimple artinian algebra $F_1 \oplus \cdots \oplus F_t$, F_i a field, $1 \leqslant i \leqslant t$, then R is a left and right order in its algebra of central quotients, which is semisimple artinian of the form $S_1 \oplus \cdots \oplus S_t$, S_i simple with center F_i.*

(ii) *If C satisfies the ascending chain condition on annihilators in C then R is a left and right order in its algebra of central quotients, which is semisimple artinian. In particular R is left and right Goldie* (*cf.* [13, p. 263]).

(iii) *If R satisfies the ascending chain condition on annihilators in R then C satisfies the ascending chain conditions on annihilators in C, implying R is left and right Goldie and is a left and right order in its semisimple artinian algebra of central quotients.*

Proof. (i) Let R' be the algebra of central quotients of R. By Proposition 5.2, cent $R' = F_1 \oplus \cdots \oplus F_t$. Let e_i be the multiplicative unit of F_i. Clearly $e_i R'$ is a semiprime P.I.-algebra with multiplicative unit e_i, and $F_i =$ cent $e_i R'$. Hence $e_i R'$ is a simple algebra S_i, and $R' = S_1 \oplus \cdots \oplus S_t$.

(ii) Immediate from (i), since C is an order in a semisimple artinian algebra.

(iii) Suppose $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_i \subseteq \cdots$ is an infinite chain of ideals of C, and suppose $B_i = \operatorname{Ann}_C A_i$, A_i ideals in C. Then $B_i = \operatorname{Ann}_C(\operatorname{Ann}_C B_i)$, so we may assume $A_i = \operatorname{Ann}_C B_i$, implying $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_i \supseteq \cdots$. Now $B_i = \operatorname{Ann}_C A_i R$ by Lemma 5.3(i). Since $\operatorname{Ann} A_1 R \subseteq \operatorname{Ann} A_2 R \subseteq \cdots$ is an ascending sequence of annihilators in R, this sequence has a maximal element $\operatorname{Ann} A_k R$ for some $k \geqslant 1$. Hence $B_1 \subseteq B_2 \subseteq \cdots$ has a maximal element B_k, so C satisfies the ascending chain condition on annihilators in C. The rest of the theorem follows from (ii). Q.E.D.

Theorem 5.4 is a nice strengthening of Goldie's theorem for semiprime

P.I.-algebras (cf. Small [20, Proposition 1.13]) and leads one to ask the following questions:

(1) Is the algebra of central quotients of a semiprime P.I.-algebra semisimple?

(2) Is the algebra of central quotients the classical algebra of quotients?

(3) Do all semiprime P.I.-algebras (with 1) have classical algebras of quotients?

The answer to each question is "no," as seen by the following examples (Bergman [9] has also, concurrently, answered question # 3 in the negative).

EXAMPLE 5.5. Let F be a field, for each j let $F[[X_j]]$ be the algebra of power series over F, and let $F[[X_j]]_0$ be the set of elements of $F[[X_j]]$ with constant term 0. Let R_0 be the direct sum $\bigoplus F[[X_j]]_0$, $1 \leqslant j < \infty$, and let R be the F-algebra formed by adjoining 1 to R_0. R is commutative and semiprime, having no nonzero nilpotent elements. On the other hand, a typical element

$$r = \alpha_0 + \sum_{i \geqslant 1} \alpha_{1i} X_1{}^i + \sum_{i \geqslant 1} \alpha_{2i} X_2{}^i + \cdots + \sum_{i \geqslant 1} \alpha_{mi} X_m{}^i, \qquad m < \infty,$$

is regular if and only if $\alpha_0 \neq 0$, in which case r is invertible in R. Hence R is its own algebra of central quotients and has the unique maximal ideal R_0 (so R is not semisimple).

The next two examples involve the Ore condition for a ring to have a classical ring of quotients, readily extendible to algebras. Any algebra has a classical algebra of left quotients if and only if it is left *Ore*, by which is meant for any r, s in R, r regular, there exist r', s' in R, r' regular, such that $s'r = r's$.

EXAMPLE 5.6. Let F be a field and let

$$X_i = \begin{pmatrix} \xi_{1i} & \xi_{2i} \\ \xi_{3i} & \xi_{4i} \end{pmatrix}, \qquad i \geqslant 1,$$

be a "generic" 2×2 matrix whose entries are commutative indeterminates over F; let $H = F[\xi]$ be the domain generated over F by the $\xi_{\mu i}$, $1 \leqslant \mu \leqslant 4$, $i \geqslant 1$, and let H' be the quotient field of H. The complete direct product $\prod M_2(H')$ of a countable number of copies of H' has multiplicative unit $1 = (1, 1, 1, \ldots)$ and is an F-algebra in the natural way. Let R be the F-subalgebra of $\prod M_2(H')$ generated by 1, $x = (X_1, X_2, X_3, \ldots)$, and the countably infinite direct *sum* $\bigoplus M_2(H')$. R is its own algebra of central quotients, is semisimple, and is left and right Ore, but the regular element x does not have an inverse in R (implying that R does not equal its classical algebra of left (or right) quotients).

Proof. Obviously R is semisimple, being the subdirect product of copies of $M_2(H')$. Note that any element in R is uniquely expressed in the form $p(x) + r$ where p is a polynomial with coefficients in F and $r \in \bigoplus M_2(H')$, and if $p(x) + r$ is regular then $p \neq 0$. Hence cent $R = F \cdot 1 + (\bigoplus H')$, implying R is its own algebra of central quotients.

Now we verify the left Ore condition. Let $a = p_1(x) + r_1$, $b = p_2(x) + r_2$, a regular. We claim there exist a', b' in R, a' regular, such that $a'b = b'a$. For any r in R, write $r = (r(1), r(2),...)$, and let $I = \{i \in \mathbb{Z}^+ \mid r_1(i) \neq 0$ or $r_2(i) \neq 0\}$. Clearly I is a finite set.

For $i \notin I$, $(p_2(x)a)(i) = p_2(X_i)\,p_1(X_i) = (bp_1(x))(i)$. Let us choose r_1', r_2' in $\bigoplus M_2(H)$ such that $r_1'(i) = r_2'(i) = 0$ for $i \notin I$, and for $i \in I$, $p_1(X_i) + r_1'(i)$ is regular in $M_2(H)$ and $(p_1(X_i) + r_1'(i))b = (p_2(X_i) + r_2'(i))a$. Obviously $a' = p_1(x) + r_1'$ is regular, and for $b' = p_2(x) + r_2'$ we have $a'b = b'a$, so R is left Ore. By symmetry R is right Ore.

Clearly x is regular since X_i is generic. If x were invertible then we would have $(p(x) + r)x = 1$ for some r in $\bigoplus M_2(H')$. Then for i such that $r(i) = 0$, $p(X_i)X_i = 1$, an absurdity. Hence x is not invertible.

EXAMPLE 5.7. *Let x, H be as in Example 5.6, and let R be the F-subalgebra of $\prod M_2(H)$ generated by* 1, x, $x' = \{X_2, 0, X_4, 0,...\}$, *and* $\bigoplus M_2(H)$. *Then R is semiprime but neither left nor right Ore.*

Proof. Since $\bigoplus M_2(H) \subset R \subset \prod M_2(H)$, R is a subdirect product of prime algebras and is thus semiprime. We claim R is not left Ore, for otherwise there exist a, b, a regular, such that $ax' = bx$. Now any element in R is of the form $p(x) + q(x, x') + r$ where p is a polynomial in one indeterminate (with coefficients in F), q is a polynomial in two indeterminates such that the second indeterminate has positive degree in each monomial of q, and $r \in \bigoplus M_2(H)$. Let $a = p_1(x) + q_1(x, x') + r_1$ and $b = p_2(x) + q_2(x, x') + r_2$. Then $p_1(x)x' + q_1(x, x')x' + r_1x' = p_2(x)x + q_2(x, x')x + r_2x$. Since $r_1, r_2 \in \bigoplus M_2(H)$, there exists $k > 0$ such that $r_1(i) = r_2(i) = 0$ for $i > k$. But x' has value 0 on all even components; evaluating both sides at the component $(2k)$ yields $0 = p_2(X_{2k})X_{2k}$. Since X_{2k} is generic, $p_2 = 0$.

Hence $p_1(x)x' + q_1(x, x')x' + r_1x' = q_2(x, x')x + r_2x$. Evaluating at the component $(2k + 1)$ yields $p_1(X_{2k+1})X_{2k+2} + q_1(X_{2k+1}, X_{2k+2})X_{2k+2} = q_2(X_{2k+1}, X_{2k+2})X_{2k+1}$. Now specialize X_{2k+1} to αe_{11} and X_{2k+2} to e_{12}. This gives us $p_1(\alpha)e_{11}e_{12} + q_1(\alpha e_{11}, e_{12})e_{12} = q_2(\alpha e_{11}, e_{21})e_{11}$. But e_{12} occurs in each term of q_1 and q_2, so $q_1(\alpha e_{11}, e_{12})e_{12} = q_2(\alpha e_{11}, e_{12})e_{11} = 0$. Hence $p_1(\alpha) = 0$, implying $p_1 = 0$. This implies $a = q_1(x, x') + r_1$ is not regular (since $a(2k) = 0$) and R is not left Ore; by symmetry R is not right Ore. Q.E.D.

In light of the examples, the following proposition has interest.

PROPOSITION 5.8. *If $P_1 \cap \cdots \cap P_t = 0$ for suitable prime ideals $P_1, \ldots, P_t$ of R then R is an order in its algebra of central quotients, which is semisimple artinian.*

Proof. C has a finite set of primes $\{P_1 \cap C, \ldots, P_t \cap C\}$ whose intersection is 0. By Theorem 5.4, we need only show C is an order in a semisimple artinian algebra, so the proposition has been reduced to the commutative case. Therefore, assume R is commutative and choose a set of primes $\{P_1, \ldots, P_m\}$ of minimal cardinality such that $P_1 \cap \cdots \cap P_m = 0$. Fix i, $1 \leqslant i \leqslant m$. By the minimality of m, there is for any j an element c_{ij} in R such that $c_{ij} \in P_j$ but $c_{ij} \notin P_i$. Letting $c_i = \prod_{j \neq i} c_{ij} \neq 0$, clearly $c_i \in P_j$ for all $j \neq i$. Embedding R in $R_1 \oplus \cdots \oplus R_m$, where $R_i = R/P_i$, we see that any regular element of R is regular in $R_1 \oplus \cdots \oplus R_m$. Let $S_1 \oplus \cdots \oplus S_m$ be the semisimple artinian quotient algebra of $R_1 \oplus \cdots \oplus R_m$. If R' is the algebra of quotients of R then $R' \hookrightarrow S_1 \oplus \cdots \oplus S_m$. But $c = \sum_{i=1}^m c_i$ is regular, so $(0, \ldots, 0, 1, 0, \ldots, 0) = c_i c^{-1} \in R'$. It follows that $R_1 \oplus \cdots \oplus R_m \subseteq R'$, so $R' \approx S_1 \oplus \cdots \oplus S_m$. Q.E.D.

Now let "Rad" denote the Jacobson radical. $C \cap \operatorname{Rad} R$ is easily seen to be a quasiregular ideal of C, so $C \cap \operatorname{Rad} R \subseteq \operatorname{Rad} C$. In particular, by Theorem 2.10, R is semiprimitive if C is semiprimitive. Conversely, we have the following theorem.

THEOREM 5.9. *The center of a semiprimitive P.I.-ring is semiprimitive.*

Proof. Suppose not. Let R be a semiprimitive P.I.-ring (i.e. set $\Omega = \mathbb{Z}$) of minimal degree n such that $\operatorname{Rad} C \neq 0$. Clearly R is not commutative, so $n > 1$. Let $J = \operatorname{Rad} C$, choose nonzero c in J, and set $A = cR$. Viewing A as ring without 1, let $C' = \operatorname{cent} A$. Clearly $C' \supseteq C \cap A$. We claim in fact $C' = C \cap A$. Indeed, for cr in C', r_1 in R, we have (where $[xy] \equiv xy - yx$) $([cr, r_1]R)^2 = c[r, r_1]R\, c[r, r_1]R \subseteq c^2[r, r_1]R = [cr, cr_1]R = 0$, so $[cr, r_1] = 0$ since R is semiprime, implying $cr \in C$; hence $C' \subseteq C \cap A$, establishing the claim.

For x in $J \cap A$, let y in C be the right quasiinverse for x. Clearly $y = xy - x \in J \cap A$, implying $J \cap A$ is a quasiregular ideal of C'. Hence $0 \neq c \in J \cap A \subseteq \operatorname{Rad} C'$.

Let $\mathscr{P} = \{\text{maximal ideals } P \text{ of } R \mid A \nsubseteq P\}$. For P in $\mathscr{P}$, $(A + P)/P$ is a nonzero ideal of the simple ring R/P, so $R/P = (A + P)/P \approx A/(A \cap P)$, implying $(A \cap P)$ is maximal in A. But

$$\bigcap\{A \cap P \mid P \in \mathscr{P}\} = A \cap \left(\bigcap\{P \mid P \in \mathscr{P}\}\right) = 0$$

(since R is semisimple). In particular, $\operatorname{Rad} A = 0$.

On the other hand, no P in $\mathscr{P}$ can be identity-faithful, because otherwise $P \cap C$ is maximal in C by Theorem 4.16(d), implying $A = cR \subseteq (P \cap C)R \subseteq P$, contrary to $P \in \mathscr{P}$. Thus, for each P in $\mathscr{P}$, $S_{2n-2}(X_1, \ldots, X_{2n-2})$ vanishes identically on $A/A \cap P \approx R/P$, implying $S_{2n-2}(X_1, \ldots, X_{2n-2})$ vanishes identically on A.

Let A_1 be the ring formed by adjoining 1 formally to A; i.e., A_1 has the additive group structure $\mathbb{Z} \oplus A$ with multiplication given by

$$(m_1, a_1)(m_2, a_2) = (m_1 m_2, m_1 a_2 + m_2 a_1 + a_1 a_2),$$

m_1, m_2 in $\mathbb{Z}$, a_1, a_2 in A. It is easy to see (and is well known) Rad $A_1 =$ Rad $A = 0$, Rad(cent A_1) $=$ Rad $C' \neq 0$, and S_{2n-2} is an identity of A_1 because S_{2n-2} is multilinear and alternating. Hence A_1 has degree $\leqslant n - 1$, contrary to the minimality of n; this contradiction yields our theorem. Q.E.D.

6. Azumaya Algebras of Finite Rank

The main result of this section is a new proof of the Artin–Procesi theorem, by showing that any identity-faithful ring satisfying the identities of $M_n(\mathbb{Z})$ is Azumaya of finite rank. In the course of the proof some interesting properties of such rings are uncovered, such as the existence of a trace if the center satisfies a certain condition. This approach yields some results on central polynomials.

Throughout we assume $\Omega = \mathbb{Z}$, i.e., R is a ring. R is an *Azumaya* algebra over C if R is projective as an $R \otimes_C R^0$ module (where R^0 is the opposite algebra and the module multiplication is given by $(\sum r_{i1} \otimes r_{i2})r = \sum r_{i1} r r_{i2}$). This is the same as R being central separable, as defined in [11], a good general reference.

Let R be Azumaya. By [11, Corollary 3.7], there is a one-to-one correspondence between ideals A of C and ideals $\tilde{A}$ of R, given by $\tilde{A} = AR$ and $A = \tilde{A} \cap R$. If $\tilde{A}$ is an ideal of R and $A = \tilde{A} \cap C$, then $R/\tilde{A}$ is Azumaya over its center, which is $(C + \tilde{A})/\tilde{A} \approx C/A$ (cf. [11, Proposition 1.11, p. 46]).

Lemma 6.1. *Suppose R is Azumaya and C is local with unique maximal ideal P. Then PR is the unique maximal ideal of R; if the simple algebra R/PR has degree n then R is free of dimension n^2 over C.*

Proof. PR is the unique maximal ideal of R, since P is the unique maximal ideal of C.

Let $r \mapsto \bar{r}$ denote the canonical homomorphism $R \to R/PR$. Now R is a finitely generated projective C-module by [11, p. 52]; since C is local, we have R is a finitely generated free C-module. Let $r_1, \ldots, r_t$ be a basis of R over

C. First we observe $r_i \notin PR$, $1 \leqslant i \leqslant t$. Indeed, if $r_i \in PR$ then $r_i = \sum_{j=1}^{t} p_j r_j$, so $r_i(1 - p_i) = \sum_{j \neq i} p_j r_j$, contrary to $r_1, ..., r_t$ being a basis. Next we claim $\bar{r}_1, ..., \bar{r}_t$ is a basis of R/PR over $\bar{C} = (C + PR)/PR$. Indeed, suppose $\sum \bar{c}_i \bar{r}_i = 0$, $\bar{c}_i \in \bar{C}$. Then $\sum c_i r_i \in PR$. Suppose $\sum c_i r_i = \sum p_i r_i$, p_i in P. Then $\sum (c_i - p_i) r_i = 0$, so each $c_i \in P$, implying $\bar{c}_i = 0$. Hence $\bar{r}_1, ..., \bar{r}_t$ is a basis of R/PR over $\bar{C} = \text{cent}(R/PR)$, implying $t = n^2$. Q.E.D.

If n = degree of R/PR is independent of the choice of P, we say R has *rank* n^2.

DEFINITION 6.2. R is an A_n-ring if

(i) All identities of $M_n(\mathbb{Z})$ are identities of R.

(ii) S_{2n-2} is not an identity of any nonzero homomorphic image of R.

Artin [7] gave a remarkable theorem, which as modified by Procesi [16] says the following:

R is Azumaya over C of rank n^2 if and only if R is an A_n-ring.

Amitsur [5] has given a simplified proof of this theorem; using the tools developed previously we shall show how in fact this theorem is a consequence of properties of central polynomials, specifically the Formanek polynomial g_n.

First we make some observations on the statement of the theorem.

(1) For the implication to hold we need the phrase "of rank n^2" since, for any field F, $M_{r_1}(F) \oplus M_{r_2}(F)$ is Azumaya over $F \oplus F$ but is not an A_n-ring unless $r_1 = r_2$.

(2) The implication, "R Azumaya of rank $n^2 \Rightarrow R$ is A_n" is easily seen to be true, so we need show only, "Any A_n-ring is Azumaya of rank n^2."

(3) If $1 \notin R$ the statement of the theorem is void. For let $\{F_i\}$ be an infinite set of copies of fields isomorphic to F, and let $R = \bigoplus_i M_n(F_i)$ be an infinite direct sum. R is easily seen to be an A_n-ring; on the other hand, R is not even finitely generated over C.

Let I_n be the additive subgroup of C generated by all values taken by the Formanek polynomial g_n. Since g_n is regular, I_n is an ideal of C contained in the central kernel I of R. The flavor of our approach is in the following.

PROPOSITION 6.3. *The following conditions are equivalent on a ring R satisfying the identities of $M_n(\mathbb{Z})$:*

(1) *R is an A_n-ring.*

(2) *g_n is central for all nonzero homomorphic images of R.*

(3) $I_n = C$.

Proof. (1) $\Rightarrow$ (2). If g_n is an identity of $R/\tilde{A}$ for some proper ideal $\tilde{A}$ then g_n is an identity of $R/\tilde{P}$ for any maximal $\tilde{P}$ containing $\tilde{A}$, implying the simple ring $R/\tilde{P}$ has degree $\leqslant n-1$, contradicting (1).

(2) $\Rightarrow$ (3). Since g_n is an identity of R/I_nR, we have $R = I_nR$ by (2). Hence $R = IR$ (because $I_n \subseteq I$), so $I = C$ by Corollary 4.20. If $I_n \neq C$ then we can embed I_n in a maximal ideal P of C, and $PR \supseteq I_nR = R$, contrary to the conclusion of Theorem 4.16(d). Thus $I_n = C$.

(3) $\Rightarrow$ (2). Since $I_n = C$, $1 \in I_n$. Hence g_n is not an identity of any nonzero homomorphic image of R.

(2) $\Rightarrow$ (1). Immediate since g_n is an identity of $M_{n-1}(\mathbb{Z})$. Q.E.D.

Suppose R is an A_n-ring and let S be a submonoid of C. By theorem 3.1 R_S satisfies the identities of $M_n(\mathbb{Z})$ and, by Proposition 6.3(3), $1 \in I_n$, so it is evident R_S is also an A_n-ring.

Let us call R *admissible* if C contains a multiplicative subgroup G whose image under any nonzero homomorphism of R contains at least n^2 elements.

For any A_n-ring R, let H be the subring of C generated by 1. Choose two distinct primes p_1 and p_2, each greater than n^2, let ξ be a primitive (p_1p_2)th root of unity, and let $R' = R \otimes_H H[\xi]$. The following result is due in the prime case to Procesi.

LEMMA 6.4. *R' is an admissible A_n-ring. If R' is Azumaya then R is Azumaya of rank n^2.*

Proof. R' satisfies the identities of $M_n(\mathbb{Z})$ by Proposition 1.3. On the other hand, Proposition 6.3(3) says there exist $k \geqslant 1$, $r_{i1},\ldots, r_{i,n+1}$ in R for $1 \leqslant i \leqslant k$, such that $\sum_{i=1}^{k} g_n(r_{i1},\ldots, r_{i,n+1}) = 1$. Hence

$$\sum_{i=1}^{k} g_n(r_{i1} \otimes 1,\ldots, r_{i,n+1} \otimes 1) = 1 \otimes 1,$$

so R' is an A_n-ring by Proposition 6.3(3). As Procesi observed, setting $G = \langle \xi \rangle$ shows R' is admissible. Now suppose R' is Azumaya. Then R is Azumaya by [11, p. 44]. Let P be a prime ideal of C. R_P is an A_n-ring, and $C_P = \operatorname{cent} R_P$ by Theorem 4.16(a); hence $R_P \approx R \otimes_C C_P$ is central separable (i.e., Azumaya) by [11, Corollary 1.7, p. 44]. By Lemma 6.1, we see R_P has dimension n^2 over its center; hence R has rank n^2, as claimed. Q.E.D.

It is immediate from Lemma 6.4 that the Artin–Procesi theorem would follow from the statement, "All admissible A_n-rings are Azumaya," which we aim to prove. Let f_n be Amitsur's polynomial of Theorem 2.7, which gives a characteristic equation in $M_n(\mathbb{Z})$ for any r_1 such that $g_n(r_1, ..., r_{n+1}) \neq 0$. Clearly f_n is an identity of any ring satisfying the identities of $M_n(\mathbb{Z})$; in particular f_n is an identity of the A_n-ring R. Assume for the remainder of this proof that R is admissible, and let G be the multiplicative subgroup of C whose image under each nonzero homomorphism of R has at least n^2 elements. We shall use f_n to define a trace on R, in terms of the central polynomials g_n and g_{1n} of Theorem 2.7.

Let $\{P_u\}$ be the set of maximal ideals of C. Since $I = C$, Theorem 4.16(d) gives us a one-to-one correspondence $P_u \to \tilde{P}_u$ into the set of maximal ideals of R, and $P_u = \tilde{P}_u \cap C$. Therefore, by Proposition 6.3, for each P_u we can find $r_{u1}, ..., r_{u,n+1}$ in R such that $g_n(r_{u1}, ..., r_{u,n+1}) \notin \tilde{P}_u$, so

$$g_n(r_{u1}, ..., r_{u,n+1}) \notin P_u .$$

Fix r in R, and for any λ in $G \cup \{0\}$ define

$$h_{0u}(\lambda) = g_n(\lambda r + r_{u1}, r_{u2}, ..., r_{u,n+1})$$

and

$$h_{1u}(\lambda) = g_{1n}(\lambda r + r_{u1}, r_{u2}, ..., r_{u,n+1}).$$

We see that $h_{0u}(0) = g_n(r_{u1}, ..., r_{u,n+1}) \notin P$.

LEMMA 6.5. *For each P_u, there exists λ_u in G such that $h_{0u}(\lambda_u) \notin P_u$.*

Proof. Viewing $h_{0u}(\lambda)$ as a polynomial in λ, we have $h_{0u}(\lambda) = \sum_{i=0}^{n^2-n} w_i \lambda^i$ (the degree of g_n in the first indeterminate is $n^2 - n$). If $h_{0u}(\lambda) \in P_u$, all λ in G, then applying the standard Vandermonde determinant argument to the simple ring $R/\tilde{P}_u$ (in which $\bar{G} = (G + \tilde{P}_u)/\tilde{P}_u$ has at least n^2 distinct elements) yields $w_i \in \tilde{P}_u$ for all i. But $w_0 = h_{0u}(0) \notin \tilde{P}_u$, a contradiction. Hence there exists λ_u in G such that $h_{0u}(\lambda_u) \notin P_u$. Q.E.D.

COROLLARY 6.6. *For suitable P_{u_i}, $1 \leqslant i \leqslant t$, there exist c_i in C and λ_i in G such that $\sum_{i=1}^{t} h_{0u_i}(\lambda_i)\, h_{0u_i}(0) c_i = 1$.*

Proof. For any P_u there exists λ_u in G such that $h_{0u}(\lambda_u) \notin P_u$. Since $h_{0u}(0) \notin P_u$ and P_u is maximal, we have $h_{0u}(\lambda_u)\, h_{0u}(0) \notin P_u$. Hence the ideal of C generated by all $h_{0x}(\lambda)\, h_{0u}(0)$, all P_u, is not contained in any maximal ideal of C and must therefore contain 1. Q.E.D.

Let $\{Y_1, Y_2, ...\}$ be a set of generic $n \times n$ matrices (i.e., Y_k has entry $\xi_{ij}^{(k)}$ in the (i, j) position, where the $\xi_{ij}^{(k)}$ are distinct commutative indeterminates over $\mathbb{Z}$) of cardinality $\geqslant$ card R, and let T be the ring $\mathbb{Z}\{Y_1, Y_2, ...\}$ generated over $\mathbb{Z}$ by these generic matrices. Since R satisfies the identities of $M_n(\mathbb{Z})$, there is an epimorphism $\varphi\colon T \to R$ in which each element of R is the image of a corresponding generic matrix Y_k.

Amitsur has proved that T is a P.I.-domain, so, by Theorem 3.3, its ring of central quotients D is a division ring. Let $C' = \operatorname{cent} T$. Since g_n is central on T and since $I_n = C$, we have $\varphi(C') = C$ by Theorem 4.22(c). Let $S = \varphi^{-1}(G) \cap C'$. Obviously $\varphi(S) = G$ and S is a submonoid of C'. We view $T \subseteq T_S \subseteq D$.

THEOREM 6.7. (a) *There is an epimorphism* $\Psi\colon T_S \to R$ *given by* $\Psi(xs^{-1}) = \varphi(x)\,\varphi(s)^{-1}$ *for all* x *in* T, s *in* S; $\operatorname{cent} T_S = C'_S$, *and* $\Psi(C'_S) = C$.

(b) *The reduced trace map* $\operatorname{tr}\colon D \to \operatorname{cent} D$, *when restricted to* T_S, *yields the map* $\operatorname{tr}\colon T_S \to C'_S$.

(c) *Define* $\operatorname{tr}\colon R \to C$ *by* $\operatorname{tr}(\Psi(x)) = \Psi(\operatorname{tr} x)$, x *in* T_S. *This is a well defined function satisfying the properties* (i) $\operatorname{tr}(r_1 + r_2) = \operatorname{tr} r_1 + \operatorname{tr} r_2$ *for all* r_1, r_2 *in* R; (ii) $\operatorname{tr}(r_1 r_2) = \operatorname{tr}(r_2 r_1)$; (iii) $\operatorname{tr}(cr) = c \operatorname{tr} r$ *for* r *in* R, c *in* C; (iv) $\operatorname{tr} 1 = n$.

(d) *For any* P_u *we have* $\operatorname{tr}(r + \tilde{P}_u) = \operatorname{tr} r + \tilde{P}_u$ *in* $R/\tilde{P}_u$, *for all* r *in* R.

Proof. (a) Immediate (cf. Section 3).

(b) To show $\operatorname{tr}\colon T_S \to C'_S$ it clearly suffices to show $\operatorname{tr}\colon T \to C'_S$. Choose $r'_{u1}, \ldots, r'_{u,n+1}$ in T for each u, such that $\varphi(r'_{uj}) = r_{uj}$, $1 \leqslant j \leqslant n+1$. Let us fix x in T and $r = \varphi(x)$ in R. For all s in $S \cup \{0\}$, let $h'_{1u}(s) = g_{1n}(sx + r'_{u1}, r'_{u2}, \ldots, r'_{u,n+1})$ and $h'_{0u}(s) = g_n(sx + r'_{u1}, r'_{u2}, \ldots, r'_{u,n+1})$. Clearly $\varphi(h'_{1u}(s)) = h_{1u}(\varphi(s))$ and $\varphi(h'_{0u}(s)) = h_{0u}(\varphi(s))$, all s in $S \cup \{0\}$.

Now by Theorem 2.7, $h'_{0u}(s) \operatorname{tr}(sx + r'_{u1}) = -h'_{1u}(s)$, all s in S, and $h'_{0u}(0) \operatorname{tr} r'_{u1} = -h'_{1u}(0)$. Thus, $h'_{0u}(s)\, h'_{0u}(0) \operatorname{tr} x = h'_{0u}(s)\, h'_{0u}(0)(\operatorname{tr}(sx + r'_{u1}) - \operatorname{tr} r'_{u1})s^{-1} = (h'_{0u}(s)\, h'_{1u}(0) - h'_{0u}(0)\, h'_{1u}(s))s^{-1}$. Let us choose c_i' in C', s_i in S such that $\varphi(c_i') = c_i$, $\varphi(s_i) = \lambda_i$, for c_i and λ_i as in Corollary 6.6, $1 \leqslant i \leqslant t$. Then

$$\varphi\left(\sum_{i=1}^{t} h'_{0u_i}(s_i)\, h'_{0u_i}(0)\, c_i'\right) = \sum_{i=1}^{t} h_{0u_i}(\lambda_i)\, h_{0u_i}(0)\, c_i = 1,$$

so $\sum_{i=1}^{t} h'_{0u_i}(s_i)\, h'_{0u_i}(0) c_i'$ is an element s_0 of $\varphi^{-1}(G) \cap C' = S$. Moreover,

$$\begin{aligned} s_0 \operatorname{tr} x &= \sum_{i=1}^{t} h'_{0u_i}(s_i)\, h'_{0u_i}(0)\, c_i' \operatorname{tr} x \\ &= \sum_{i=1}^{t} (h'_{0u_i}(s_i)\, h'_{1u_i}(0) - h'_{0u_i}(0)\, h'_{1u_i}(s_i))\, c_i' s_i^{-1} \in T_S, \qquad (1) \end{aligned}$$

so $\operatorname{tr} x \in T_S$.

(c) Since tr is linear on D, hence on T_S, we will know that tr: $R \to C$ is well defined once we have shown that for all y in $\Psi^{-1}(0) \subseteq T_S$, $\Psi(\operatorname{tr} y) = 0$. Suppose $y = xs^{-1}$, x in T, s in S. Then $r = \varphi(x) = 0$, so $h_{1u_i}(\lambda_i) = h_{1u_i}(0)$ and $h_{0u_i}(\lambda_i) = h_{0u_i}(0)$, all i. Applying Ψ to both sides of (1) yields

$$\Psi(\operatorname{tr} x) = \sum_{i=1}^{t} (h_{0u_i}(\lambda_i)\, h_{1u_i}(0) - h_{0u_i}(0)\, h_{1u_i}(\lambda_i))\, c_i\lambda_i^{-1} = 0,$$

so $\Psi(\operatorname{tr} y) = \Psi(\operatorname{tr} x)\,\varphi(s)^{-1} = 0$. Therefore tr: $R \to C$ is well defined and properties (i)–(iv) are inherited from tr on T_S.

(d) For any element r of R, let $\bar{r}$ denote $r + \tilde{P}_u$ in $R/\tilde{P}_u$. Thus we want $\operatorname{tr} \bar{r} = \overline{\operatorname{tr} r}$, all r in R. Fix r and choose x in T, s_u in S such that $\varphi(x) = r$, $\varphi(s_u) = \lambda_u$, λ_u as in Lemma 6.5. Defining $r'_{u1},\ldots, r'_{u,n+1}$, h'_{1u}, h'_{0u} as in (b), we have, from Theorem 2.7,

$$h'_{0u}(s_u)\operatorname{tr}(s_u x + r'_{u1}) = -h'_{1u}(s_u) \qquad \text{and} \qquad h'_{0u}(0)\operatorname{tr} r'_{u1} = -h'_{1u}(0).$$

Applying Ψ to both sides of each equation yields $h_{0u}(\lambda_u)\operatorname{tr}(\lambda_u r + r_{u1}) = -h_{1u}(\lambda_u)$ and $h_{0u}(0)\operatorname{tr} r_{u1} = -h_{1u}(0)$. Taking images in $R/\tilde{P}_u$, we have

$$\overline{h_{0u}(\lambda_u)\operatorname{tr}(\lambda_u r + r_{u1})} = -\overline{h_{1u}(\lambda_u)} \qquad \text{and} \qquad \overline{h_{0u}(0)}\,\overline{\operatorname{tr} r_{u1}} = -\overline{h_{1u}(0)}.$$

But Theorem 2.7 applied to $R/\tilde{P}_u$ yields $\overline{h_{0u}(\lambda_u)}\operatorname{tr}(\bar{\lambda}_u\bar{r} + \overline{r_{u1}}) = -\overline{h_{1u}(\lambda_u)}$ and $\overline{h_{0u}(0)}\operatorname{tr}\overline{r_{u1}} = -\overline{h_{1u}(0)}$. Since $\overline{h_{0u}(\lambda_u)} \neq 0$ and $\overline{h_{0u}(0)} \neq 0$ (by definition of λ_u in Lemma 6.5), we conclude $\overline{\operatorname{tr}(\lambda_u r + r_{u1})} = \operatorname{tr}(\bar{\lambda}_u\bar{r} + \overline{r_{u1}})$ and $\overline{\operatorname{tr} r_{u1}} = \operatorname{tr}\overline{r_{u1}}$. Hence $\overline{\operatorname{tr}\lambda_u r} = \operatorname{tr}\overline{\lambda_u r}$. But $\bar{\lambda}_u \neq 0$, so $\overline{\operatorname{tr} r} = \operatorname{tr}\bar{r}$, as desired.

Q.E.D.

It is easy now to show that R is Azumaya. Let P be any maximal ideal of C and let $\tilde{P}$ be the maximal ideal of R containing P. $R/\tilde{P}$ is simple with center isomorphic to C/P (by Theorem 4.22). Moreover, $R/\tilde{P}$ is of degree n since R is an A_n-ring, so $R/\tilde{P}$ has dimension n^2 as C/P-module, having base $\bar{r}_{n1},\ldots, \bar{r}_{n^2}$. Let $(\operatorname{tr}(r_i r_j))$ be the $n^2 \times n^2$ matrix whose (i, j)-entry is $\operatorname{tr}(r_i r_j)$, $1 \leqslant i, j \leqslant n^2$, and let $d = \det(\operatorname{tr}(r_i r_j))$. Since $\operatorname{tr}(\bar{r}_i\bar{r}_j)) = \overline{(\operatorname{tr}(r_i r_j))}$, we have $\bar{d} = \det(\operatorname{tr}(\bar{r}_i\bar{r}_j)) \neq 0$. Hence $d \notin P$. Let R_d be the localization of R by the set $\{d^k \mid k = 0, 1,\ldots\}$. Artin [7, p. 553] has shown that $r_1 1^{-1},\ldots, r_{n^2}1^{-1}$ are linearly independent over cent R_d, using the following argument:

Suppose $\sum_{i=1}^{n^2} c_i r_i 1^{-1} = 0$, $c_i \in$ cent R_d for all i. Then $\sum_{i=1}^{n^2} c_i \operatorname{tr}(r_i r_j)1^{-1} = 0$. This gives us a system of n^2 equations in the c_i, $1 \leqslant i \leqslant n^2$. Solving these equations gives us $(d1^{-1})c_k = 0$ for all k, $1 \leqslant k \leqslant n^2$. Thus $c_k = 0$, $1 \leqslant k \leqslant n^2$.

Now choose $r_1',\ldots, r_{n^2}'$ in T such that $\varphi(r_i') = r_i$, $1 \leqslant i \leqslant n^2$, and let $y = \det(r_i' r_j'))$. Clearly $y \in C_S'$ and $\Psi(y) = d \neq 0$, so $y \neq 0$. Let T' be the localization of T_S by $\{y^k \mid k = 0, 1,...\}$. Viewing $T \subsetneq T_S \subsetneq T' \subseteq D$, we see (by Artin's argument given above) that $r_1',..., r'_{n^2}$ are linearly independent in T' over cent T'. Since D is the ring of central quotients of T', it follows that $r_1',..., r'_{n^2}$ are linearly independent over cent D and must be a basis for D over cent D since D has degree n. Suppose for r' in T',

$$r' = \sum_{i=1}^{n^2} c_i r_i', \qquad c_i \text{ in cent } D.$$

Then

$$\operatorname{tr}(r'r_j') = \sum_{i=1}^{n^2} c_i \operatorname{tr}(r_i'r_j'), \qquad 1 \leqslant j \leqslant n^2.$$

Since $\det(\operatorname{tr}(r_i'r_j')) = y$ is invertible in C_S', we can solve this system of equation for c_i, $1 \leqslant i \leqslant n^2$ and get $c_i \in T'$ in particular. Thus $r_1',..., t'_{n^2}$ is a basis for T' over cent T'. But there is an epimorphism of R' onto R_d given by $xy^{-k} \mapsto \Psi(x)d^{-k}$ for x in T_S, so $r_1,..., r_{n^2}$ is a basis for R_d over cent R_d. Thus we have found $d \notin P$ such that R_d is free of dimension n^2 over cent R_d. It follows by Theorem 1(e) of [10, p. 138] that R is finitely generated projective over C. Thus R is finitely presented, so R is Azumaya by [10, p. 180, Example 14(a)].

Therefore all admissible A_n-rings are Azumaya, which concludes the proof of the Artin–Procesi theorem. Q.E.D.

Since our proof is largely dependent on ideas of Procesi [16], it might be worthwhile to sketch his proof briefly, indicating similarities and differences between his proof and this proof. Procesi defines a trace on prime local admissible A_n-rings R. This trace has the property $\operatorname{tr} \bar{r} = \overline{\operatorname{tr} r}$, where $\bar{r}$ is the image in $R/\tilde{M}$ of an element r of R, $\tilde{M}$ the unique maximal ideal of R. This trace is used to prove that prime local admissible A_n-rings are Azumaya [16, Lemma 3.5], implying prime admissible A_n-rings are Azumaya [16, Lemma 3.7]. Using a result by Small on noncentral localization in P.I.-rings [16, Theorem 2.2], he shows that it is enough to consider localizations of T which are A_n-rings, T defined as in the proof given here. Now T is prime, and by adjoining a primitive p_1p_2th root of unity, $p_1, p_2 > n^2$, he makes the localizations of T into prime admissible A_n-rings, reducing the theorem to a case he has already handled.

The proof presented in this paper follows similar lines. Using *central* localization, an easy notion to grasp, we find that the concept of A_n-ring is strongly connected to Formanek's polynomial (Proposition 6.3). Amitsur's "generic" characteristic equation (Theorem 2.7) then permits us to define a trace on admissible A_n-rings, which depends essentially on various central polynomials and which therefore is inherited in homomorphic images. With this trace, we can use the discriminant argument of Artin (which Procesi also needs) to show admissible A_n-rings are Azumaya. Then it follows easily that all A_n-rings are Azumaya, of rank n^2. As presented here, the crucial part of the proof is Theorem 4.16. Since we do not need $\tilde{P} = PR$ in part (d), the proof does not rely on Theorem 4.6, the only result in Section 4 which is not derived entirely from elementary properties of central localization.

Proposition 6.3 and the Artin–Procesi theorem combine to give interesting insights about Azumaya algebras. In the following discussion, let R satisfy the identities of $M_n(Z)$ and let I' be the additive subgroup of C generated by all values taken by central polynomials of $M_n(\mathbb{Z})$. I' is a submonoid of C.

LEMMA 6.8. *R is an A_n-ring if and only if $1 \in I'$.*

Proof. If R is an A_n-ring then $1 \in I_n$ by Proposition 6.3(3), so clearly $1 \in I'$. Conversely, suppose $1 \in I'$. Suppose a homomorphic image $\bar{R}$ of R has degree $\leqslant n - 1$. Then all central polynomials of $M_n(\mathbb{Z})$ are identities of $\bar{R}$, so $\bar{R} = 0$. This shows R is an A_n-ring, by definition. Q.E.D.

Note that if R is prime, then R satisfies the identities of $M_n(F)$ for some field F, and the same argument as in Lemma 6.8 shows that R is A_n if and only if $1 \in I$, the central kernel of R.

COROLLARY 6.9. *If C is a domain then I'/I_n is nil.*

Proof. Choose $y \in I'$ and let $S = \{y^k \mid k = 0, 1,...\}$. As observed following Proposition 6.3, R_S is an A_n-ring. Hence $cy^{-j} = 1 \cdot 1^{-1}$ for some c in I_n, $j \geqslant 0$. But C is a domain, so $c = y^j$. Q.E.D.

If R is taken to be $\mathbb{Z}\{Y\}$, the ring of generic $n \times n$ matrices, Corollary 6.9 shows that any central polynomial of $M_n(\mathbb{Z})$ has a power in the T-ideal generated by g_n.

We conclude this section with a general way of constructing Azumaya algebras of rank n^2. In this construction one could substitute throughout I_n for I', since $1 \in I_n \Leftrightarrow 1 \in I'$ for any ring satisfying the identities of $M_n(\mathbb{Z})$, by Lemma 6.8.

Let $T = \mathbb{Z}\{Y_1, Y_2,...\}$, generated by a set of generic matrices of arbitrary infinite cardinality, let $C' = \text{cent } T$, and let $S_0 = (C' - \mathbb{Z}) \cup \{1\}$, a submonoid of C'.

To construct an Azumaya algebra, we let $S \neq \{1\}$ be a submonoid of S_0 and let R be a nonzero homomorphic image of T_S. Clearly $1 \in I'$, so R is an A_n-ring; hence R is Azumaya of rank n^2.

Conversely, suppose R is an A_n-ring. If we form T with a set of generic matrices of cardinality $\geqslant$ card R then there is an epimorphism φ: $T \to R$.

Since $I_n = C$, we have, by Theorem 4.22, $\varphi(C') = C$. Let $S = \varphi^{-1}(1) \cap S_0$. Obviously $S \neq 1$ is a submonoid of S_0, and there is a canonical homomorphism Ψ: $T_S \to R$ given by $\Psi(xs^{-1}) = \varphi(x)$ for all x in T, s in S. This shows our construction is general.

Let R be an A_n-ring which is not a division algebra. Then T_S, as defined in the preceding paragraph, is not a division algebra, so there is a central polynomial $g(X_1, ..., X_m)$ such that $S \cap g(Y_1, ..., Y_m)T = \varnothing$.

References

1. S. A. Amitsur, Associative rings with identities, "Some Aspects of Ring Theory," C.I.M.E. Conference, II Ciclo, August 23–31, 1965, pp. 1–48.
2. S. A. Amitsur, Prime rings having polynomial identities with arbitrary coefficients, *Proc. London Math.* **17** (1967), 470–486.
3. S. A. Amitsur, Rings with involutions, *Israel J. Math.* **6** (1968), 99–106.
4. S. A. Amitsur, A note on P.I.-rings, *Israel J. Math.* **10** (1971), 210–211.
5. S. A. Amitsur, Polynomial identities and Azumaya algebras, *J. Algebra*, to appear.
6. S. A. Amitsur and J. Levitzki, Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1** (1950), 449–463.
7. M. Artin, On Azumaya algebras and finite dimensional representations of rings, *J. Algebra* **11** (1969), 532–563.
8. G. Azumaya, On maximally central algebras, *Nagoya Math. J.* **2** (1951), 119–150.
9. G. Bergman, Some examples in P.I.-ring theory *Israel J. Math.*, to appear.
10. N. Bourbaki, "Algèbre Commutative," chapter II, Hermann, Paris, 1961.
11. F. De Meyer and E. Ingraham, "Separable Algebras over Commutative Rings," Springer-Verlag, Heidelberg/Berlin/New York, 1971.
12. E. Formanek, Central polynomials for matrix rings, *Journal of Algebra* **23** (1972), to appear.
13. N. Jacobson, "Structure of Rings," American Mathematics Society Colloquium Publication XXXVII, Providence, RI, 1964.
14. N. Jacobson, "Structure and Representation of Jordan Algebras," American Mathematics Society Colloquium Publication XXXIX, Providence, RI, 1968.
15. C. Procesi, Non-commutative affine rings, *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I* **VIII** (1967).
16. C. Procesi, On a theorem of M. Artin, *J. Algebra* **22** (1972), 309–316.
17. L. Rowen, Some results on the center of a ring with polynomial identity, *Bull. Amer. Math. Soc.* (1973), to appear.
18. L. Rowen, On classical quotients of polynomial identity rings with involution, *Proc. Amer. Math. Soc.*, to appear.
19. L. Small, Localization in P.I.-rings, *J. Algebra* **18** (1971), 269–270.
20. L. Small, Orders in Artinian rings, *J. Algebra* **4** (1966), 13–41.